

Exponential Functions

Exponents Review

Recall:

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$

$$x^5 = x \times x \times x \times x \times x$$

$$y = a^x$$

← exponent
↑
base

Laws of Exponents

Name	Law	Example
Product	$a^m \cdot a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$
Quotient	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$	$3^5 \div 3^2 = \frac{3^5}{3^2} = 3^{5-2} = 3^3$
Power	$(a^m)^n = a^{m \cdot n}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$
Negative Exponent	$a^{-m} = \frac{1}{a^m}$	$3^{-5} = \frac{1}{3^5}$
Zero Exponent	$a^0 = 1$	$3^0 = 1$
Fractional Exponent	$a^{\frac{m}{n}} = \sqrt[n]{a^m}$	$3^{\frac{2}{5}} = \sqrt[5]{3^2}$
Power of a Product	$(ab)^m = a^m \cdot b^m$	$(3 \cdot 4)^5 = 3^5 \cdot 4^5$
Power of a Fraction	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$

Solving Exponents

Ex:

a)

$$5^3 = x$$

$$125 = x$$

b)

$$2^{-3} = x$$

$$\frac{1}{2^3} = x$$

$$\frac{1}{8} = x$$

c)

$$3^x = \frac{1}{27}$$

$$3^3 = 27$$

So...

$$3^{-3} = \frac{1}{27}$$

So...

$$x = -3$$

d)

$$7^{(5-x)} = \frac{1}{49}$$

$$7^2 = 49$$

So...

$$7^{-2} = \frac{1}{49}$$

So...

$$-2 = 5 - x$$

$$-7 = -x$$

$$7 = x$$

e)

$$x^6 = 64$$

$$x = \sqrt[6]{64}$$

$$x = 2$$

f)

$$5^{2x} = 625$$

$$5^4 = 625$$

So...

$$4 = 2x$$

$$2 = x$$

Simplify Using Laws of Exponents

a)

$$\begin{aligned} & \left(\frac{x^5}{x^3}\right)(x^{-1}) \\ &= (x^2)(x^{-1}) \\ &= x \end{aligned}$$

b)

$$\begin{aligned} & 6\left(\frac{a^2}{b^6}\right)\left(\frac{b^{-2}}{a^4}\right) \\ &= 6\left(\frac{a^2b^{-2}}{b^6a^4}\right) \\ &= 6(a^{-2})(b^{-8}) \\ &= \frac{6}{a^2b^8} \end{aligned}$$

c)

$$\begin{aligned} & x^7\left(\frac{x^{-2}}{x^{-5}}\right)^3 \\ &= x^7(x^3)^3 \\ &= x^7(x)^9 \\ &= x^{16} \end{aligned}$$

d)

$$\begin{aligned} & (-3a)^4(a^4b^7) \\ &= (-3)^4a^4(a^4b^7) \\ &= 81a^8b^7 \end{aligned}$$

e)

$$\begin{aligned} & \frac{(4x^2)(xy^3)^2}{2x^4y^6z^{-3}} \\ &= \frac{(4x^2)x^2y^6}{2x^4y^6z^{-3}} \\ &= \frac{4x^4y^6}{2x^4y^6z^{-3}} \\ &= \frac{2}{z^{-3}} \\ &= 2z^3 \end{aligned}$$

f)

$$\begin{aligned} & \sqrt{\left(\frac{25}{5^4}\right)^{-3}} \\ &= \left(\frac{25}{5^4}\right)^{-3/2} \\ &= \left(\frac{5^2}{5^4}\right)^{-3/2} \\ &= (5^{-2})^{-3/2} \\ &= 5^3 \\ &= 125 \end{aligned}$$

g)

$$\begin{aligned} & \frac{16^3 \cdot 4^2}{4^3} \\ &= \frac{(4^2)^3 \cdot 4^2}{4^3} \\ &= \frac{4^6 \cdot 4^2}{4^3} \\ &= \frac{4^8}{4^3} \\ &= 4^5 \\ &= 1024 \end{aligned}$$

h)

$$\begin{aligned} & \frac{\sqrt[3]{81} \cdot \sqrt{9} \cdot 3}{27} \\ &= \frac{\sqrt[3]{3^4} \cdot \sqrt{3^2} \cdot 3}{3^3} \\ &= \frac{3^{4/3} \cdot 3 \cdot 3}{3^3} \\ &= \frac{3^{10/3}}{3^3} \\ &= 3^{(10/3-3)} \\ &= 3^{1/3} \end{aligned}$$

Exponential Basic Function

The basic exponential function is $y = c^x$, where $c > 0$ and $c \neq 1$

The base (c) determines whether the function is increasing or decreasing

- If $c > 1$ the function is increasing
- If $0 < c < 1$ the function is decreasing

The base (c) also determines the steepness or the curve

- If $c > 1$, a larger c value leads to a steeper curve
- If $0 < c < 1$, a smaller c value leads to a steeper curve

The basic function has an asymptote at $y = 0$

Sketching Exponential Functions

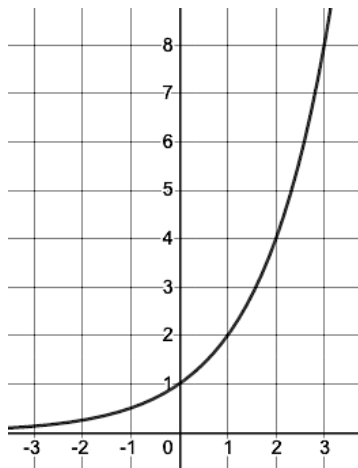
To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions

a)

$$y = 2^x$$

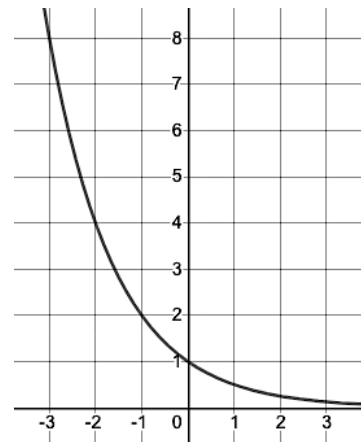
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



b)

$$y = \left(\frac{1}{2}\right)^x \text{ or } y = 2^{-x}$$

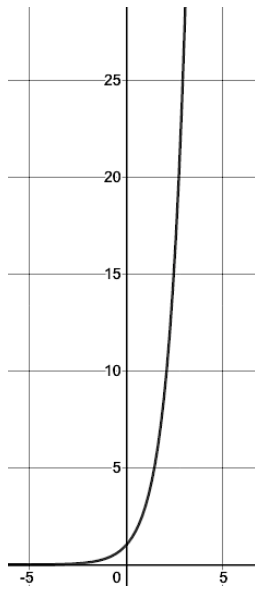
x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



c)

$$y = 3^x$$

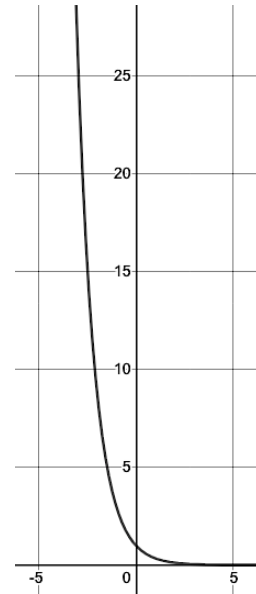
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27



d)

$$y = \left(\frac{1}{3}\right)^x \text{ or } y = 3^{-x}$$

x	y
-3	27
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$
3	$\frac{1}{27}$



Exponential Transformed Function

Just like the other functions we've looked at this year, the transformed exponential function has the parameters a , b , h , and k .

$$y = ac^{b(x-h)} + k$$

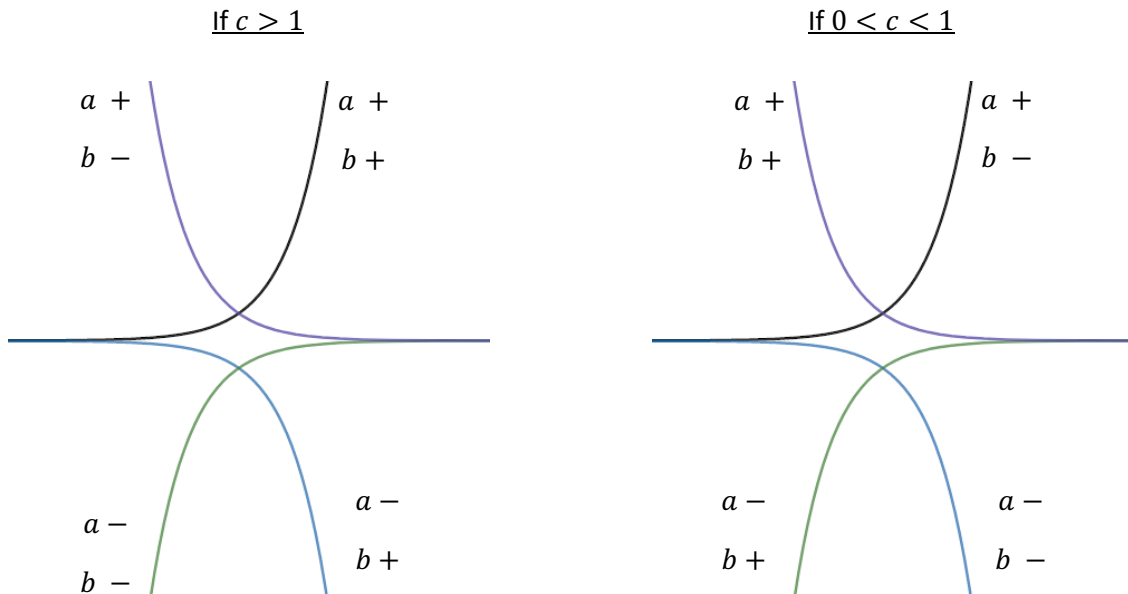
where $c \neq 1, c > 0, a \neq 0, b \neq 0$

a and b determine the direction of the curve (increasing or decreasing)

- Negative a is a reflection over the x -axis
- Negative b is a reflection over the y -axis

$y = k$ is the asymptote

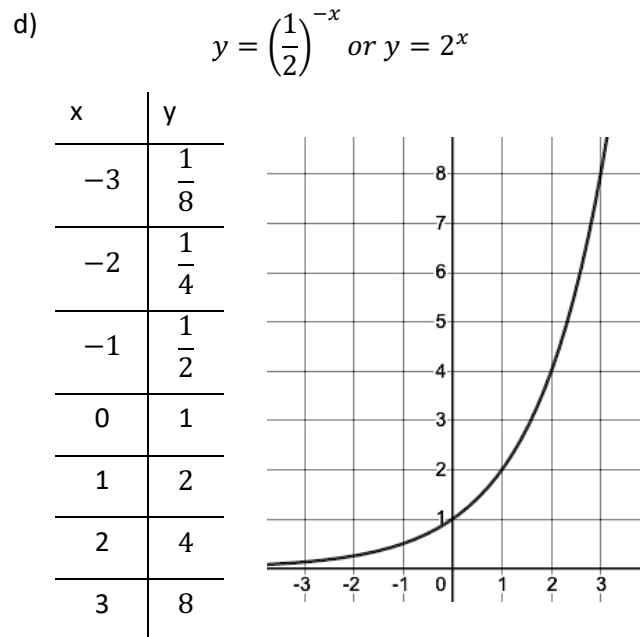
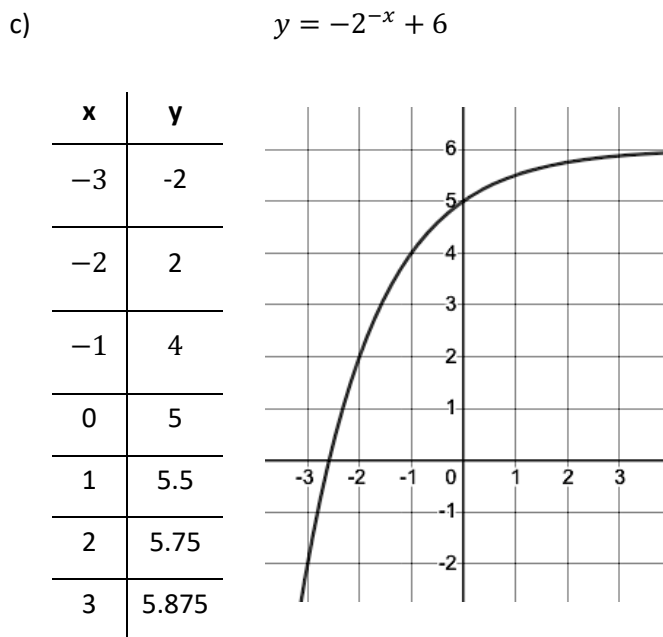
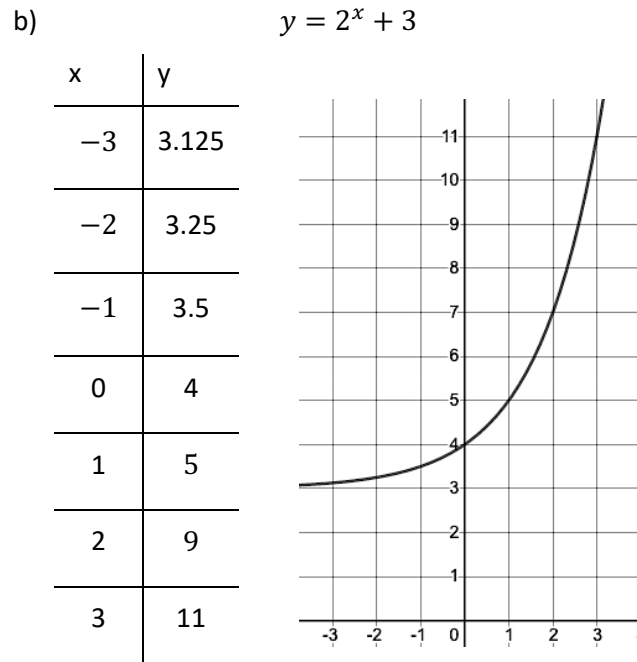
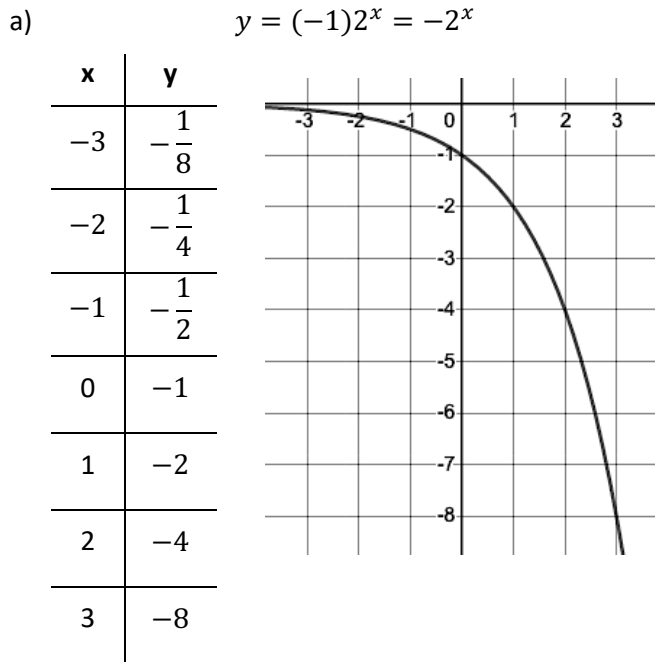
h is a vertical shift



Sketching Transformed Exponential Functions

To sketch an exponential function, create a table of values, plot the points, and connect the dots, remembering the asymptote.

Ex: Create a table of values and sketch the following functions



Finding the Rule of Exponential Functions

Recall $y = ac^{b(x-h)} + k$

Using the laws of exponents, we can re-write this as $y = ac^x + k$

Ex: Using the laws of exponents, re-write each function as $y = ac^x + k$

a) $y = 7\left(\frac{1}{2}\right)^{-2(x-2)} + 6$

$$y = 7\left(\frac{1}{2}\right)^{-2(x-2)} + 6$$

$$y = 7\left(\left(\frac{1}{2}\right)^{-2}\right)^{(x-2)} + 6$$

$$y = 7(4)^{(x-2)} + 6$$

$$y = 7(4)^{(x)} \div 4^2 + 6$$

$$y = 7 \cdot 4^x \div 16 + 6$$

$$y = 7 \cdot 4^x \cdot \frac{1}{16} + 6$$

$$y = \frac{7}{16} \cdot 4^x + 6$$

b) $y = 5(3)^{2x+2} + 7$

$$y = 5(3)^{2x+2} + 7$$

$$y = 5(3)^{2(x+1)} + 7$$

$$y = 5(3^2)^{(x+1)} + 7$$

$$y = 5(9)^{(x+1)} + 7$$

$$y = 5 \cdot 9^x \cdot 9^{(1)} + 7$$

$$= 5 \cdot 9^x \cdot 9 + 7$$

$$= 45 \cdot 9^x + 7$$

We will use this simplified version of the rule when finding the rule.

- a is the initial value
- k is the value of the asymptote (or constant)
- c is the base (or rate of change)
 - ex: "triples" means $c = 3$; "doubles" means $c = 2$; "half" means $c = 0.5$
 - If rate of change is given as a percent:
 - If increasing, $c = 1 + \frac{\%}{100}$
 - If decreasing, $c = 1 - \frac{\%}{100}$
 - Ex: loses 30% means $c = 1 - \frac{30}{100} = 0.7$
 - Ex: increases 35% means $c = 1 + \frac{35}{100} = 1.35$
 - Ex: earns 4.25% interest means $c = 1 + \frac{4.25}{100} = 1.0425$
 - Ex: depreciates by 12% means $c = 1 - \frac{12}{100} = 0.88$

To find a rule from words, determine a , c , and k and write as a rule

a) You invest \$2000 into a savings account with an interest rate of 4.125% compounded annually.

$$a = 2000$$

$$c = 1 + \frac{4.125}{100} = 1.0415$$

$$k = 0$$

$$y = 2000^{1.0415}$$

b) 200 bacteria are growing in a petri dish. Every seven days the population increases by 17%.

$$a = 200$$

$$c = 1 + \frac{17}{100} = 1.17$$

$$k = 0$$

$$y = 200^{1.17}$$

c) There are currently 125 fish in a lake. The population of fish is doubling every year.

$$a = 125$$

$$c = 2$$

$$k = 0$$

$$y = 125^2$$

d) A ball is dropped from 2m above a table that is 0.75m above the floor. The ball only regains 80% of its height after each bounce.

$$a = 2$$

$$c = 1 - \frac{20}{100} = 0.8$$

$$k = 0.75$$

$$y = 2^{0.8} + 0.75$$

To find a rule given 2 points (given in a table of value, on a graph, or in a word problem) we can find the rule using a system of equations

1. Use $y = ac^x + k$
2. Plug in points to make 2 equations
3. Eliminate 1 variable by dividing (usually eliminate a)
4. Solve for c
5. Use c to solve for a
6. Write the rule

a) Find the rule of an exponential function (using $y = ac^x$) passing through the points (1, 24) and (4, 648)

Step 1

$$y = ac^x$$

Step 2

$$\text{Function 1: } 648 = ac^4$$

$$\text{Function 2: } 24 = ac^1$$

Step 3

$$648 = ac^4$$

$$\div 24 = \frac{ac^4}{ac^1}$$

$$27 = c^3$$

Step 4

$$27 = c^3$$

$$\sqrt[3]{27} = c$$

$$3 = c$$

Step 5

$$648 = ac^4$$

$$648 = a(3)^4$$

$$648 = 81a$$

$$8 = a$$

Step 6

$$y = 8(3)^x$$

b) Find the rule of an exponential function (using $y = ac^x$) passing through the points $(-2, -0.16)$ and $(4, -2500)$

Step 1

$$y = ac^x$$

Step 2

$$\text{Function 1: } -2500 = ac^4$$

$$\text{Function 2: } -0.16 = ac^{-2}$$

Step 3

$$-2500 = ac^4$$

$$\div -0.16 = ac^{-2}$$

$$15625 = c^6$$

Step 4

$$15625 = c^6$$

$$\sqrt[6]{15625} = c$$

$$5 = c$$

Step 5

$$-0.16 = ac^{-2}$$

$$-0.16 = a(5)^{-2}$$

$$-0.16 = 0.04a$$

$$-4 = a$$

Step 6

$$y = -4(5)^x$$

c) Find the rule of an exponential function passing through the points $(1, 7.4)$ and $(2, 5.78)$ with an asymptote at $y = 5$

Step 1

$$y = ac^x + k$$

Step 2

$$\text{Function 1: } 7.4 = ac^1 + 5$$

$$2.4 = ac^1$$

$$\text{Function 2: } 5.78 = ac^2 + 5$$

$$0.78 = ac^2$$

Step 3

$$0.78 = ac^2$$

$$\div 2.4 = ac^1$$

$$0.325 = c$$

Step 4

$$0.325 = c$$

Step 5

$$7.4 = ac^1 + 5$$

$$7.4 = a(0.325)^1 + 5$$

$$2.4 = 0.325a$$

$$7.38 = a$$

Step 6

$$y = 7.38(0.325)^x + 5$$

Solving Exponential Functions

To solve for y, plug in x and solve.

Ex: Solve $f(x) = 3(4)^{x-2} - 192$ when $x = 4$

$$f(x) = 3(4)^{x-2} - 192$$

$$f(4) = 3(4)^{4-2} - 192$$

$$f(4) = 3(4)^2 - 192$$

$$f(4) = 3 \cdot 16 - 192$$

$$f(4) = 48 - 192$$

$$f(4) = -144$$

To solve for x

- Isolate the base(s)
- Make the bases the same
- When the bases are the same, exponents must be equal, so solve for x
- Note: We will also solve exponentials using logarithmic functions once we learn those

Ex: Solve for x

a)

$$3(4)^{x-2} - 192 = 0$$

$$3(4)^{x-2} = 192$$

$$(4)^{x-2} = 64$$

$$(4)^{x-2} = 4^3$$

$$x - 2 = 3$$

$$x = 5$$

b)

$$625 \left(\frac{1}{5}\right)^{3x} - 1 = 0$$

$$625 \left(\frac{1}{5}\right)^{3x} = 1$$

$$\left(\frac{1}{5}\right)^{3x} = \frac{1}{625}$$

$$\left(\frac{1}{5}\right)^{3x} = \frac{1}{5^4}$$

$$\left(\frac{1}{5}\right)^{3x} = \left(\frac{1}{5}\right)^4$$

$$3x = 4$$

$$x = \frac{4}{3}$$

c)

$$11(7)^{2x-1} = 539$$

$$(7)^{2x-1} = 49$$

$$(7)^{2x-1} = 7^2$$

$$2x - 1 = 2$$

$$2x = 3$$

$$x = 1.5$$

d)

$$\left(\frac{1}{4}\right)^{8x} = 2^{-10x+18}$$

$$(2^{-2})^{8x} = 2^{-10x+18}$$

$$(2)^{-16x} = 2^{-10x+18}$$

$$-16x = -10x + 18$$

$$-6x = 18$$

$$x = -3$$

Point of intersection

e)

$$\left(\frac{1}{2}\right)^x + 4 = 0$$

There is an asymptote at $y = 4$ and the function is increasing, so it never touches 0.

No solution.

g)

$$\begin{aligned} (3^{x+4})^2 &= \left(\frac{1}{9}\right)^x \\ 3^{2x+8} &= \left(\frac{1}{9}\right)^x \\ 3^{2x+8} &= (3^{-2})^x \\ 3^{2x+8} &= (3)^{-2x} \end{aligned}$$

$$\begin{aligned} 2x + 8 &= -2x \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

i)

$$\begin{aligned} 27(9)^x &= \sqrt{3^{10}} \\ 27(9)^x &= 3^{\frac{10}{2}} \\ 3^3 \cdot (9)^x &= 3^5 \\ 3^3 \cdot (3)^{2x} &= 3^5 \\ 3^{3+2x} &= 3^5 \end{aligned}$$

$$\begin{aligned} 3 + 2x &= 5 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

f)

$$\begin{aligned} 27(9)^{x+6} - 6 &= \sqrt{3^{10}} - 6 \\ 27(9)^{x+6} &= \sqrt{3^{10}} \\ 27(9)^{x+6} &= 243 \\ (9)^{x+6} &= 9 \end{aligned}$$

$$\begin{aligned} x + 6 &= 1 \\ x &= -5 \end{aligned}$$

h)

$$\begin{aligned} 2(3)^{x-4} &= 2 \\ (3)^{x-4} &= 1 \\ (3)^{x-4} &= 3^0 \end{aligned}$$

1s are your friends!
Anything to the power of 0 is 1.

$$\begin{aligned} x - 4 &= 0 \\ x &= 4 \end{aligned}$$

Solving Exponential Inequalities

To solve exponential inequalities, change inequality to =, solve, and then consult a sketch to answer the question over the appropriate domain (or use a test point).

To use a test point:

- pick a number either larger or smaller than the solution
- Plug that into the inequality and solve
- If you get a true statement, you know the answer must include that point. If you get a false statement, you know the solution cannot include that point.
- All solutions will either be $]-\infty, x]$, $]-\infty, x[$, $[x, \infty[$, or $]x, \infty[$

Ex: Solve $234(3)^{-0.08x} - 26 > 0$

$$234(3)^{-0.08x} - 26 > 0$$

$$234(3)^{-0.08x} - 26 = 0$$

$$234(3)^{-0.08x} = 26$$

$$(3)^{-0.08x} = \frac{1}{9}$$

$$(3)^{-0.08x} = 9^{-1}$$

$$(3)^{-0.08x} = 3^{-2}$$

$$-0.08x = -2$$

$$x = 25$$

Test point

Let $x = 0$

$$234(3)^{-0.08x} - 26 > 0$$

$$234(3)^0 - 26 > 0$$

$$234(1) - 26 > 0$$

$$208 > 0$$

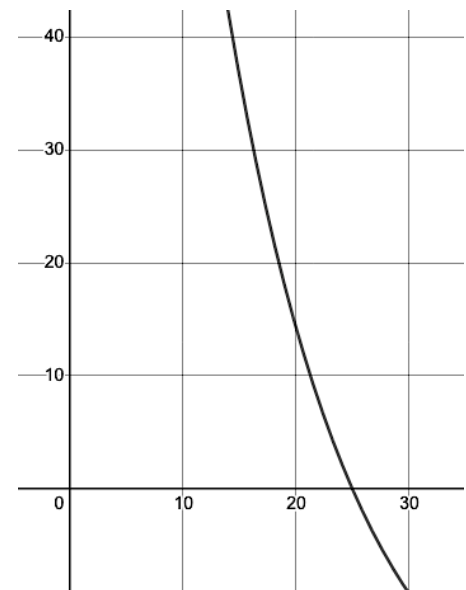
$$208 > 0$$

TRUE

So solution must include $x = 0$

$$\therefore]-\infty, 25[$$

Sketch



$$\therefore]-\infty, 25[$$

Ex: Solve $81^{x-9} \geq 1$

$$81^{x-9} = 1$$

$$81^{x-9} = 81^0$$

$$x - 9 = 0$$

$$x = 9$$

Test point

Let $x = 0$

$$81^{x-9} \geq 1$$

$$81^{-9} \geq 1$$

$$81^{x-9} \geq 1$$

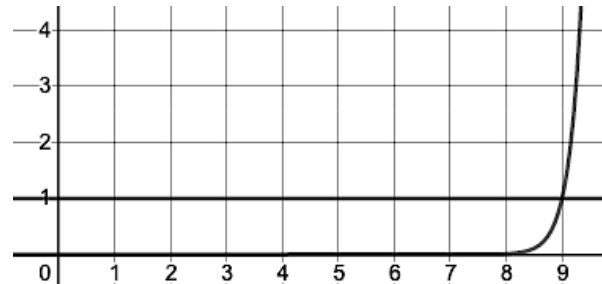
FALSE

So solution must not include

$x = 0$

$$\therefore [9, \infty[$$

Sketch



$$\therefore [9, \infty[$$

Inverse of Exponential Functions

To find the inverse of exponential functions, we will need logarithmic functions.