

Absolute Value Functions

Properties of absolute values

Property	Example
$ x \geq 0$	$ -7 = 7$ and $ 5 = 5$
$ x = -x $	$ -3 = 3 $ $3 = 3$
$ x \cdot y = x \cdot y $	$ 3 \cdot -2 = 3 \cdot -2 $ $ -6 = 3 \cdot 2$ $6 = 6$
$\left \frac{x}{y}\right = \frac{ x }{ y }$	$\left \frac{-4}{8}\right = \frac{ -4 }{ 8 }$ $ -0.5 = \frac{4}{8}$ $0.5 = 0.5$
$ x + x = 2 x $	$ 6 + 6 = 2 6 $ $6 + 6 = 2 \cdot 6$ $12 = 12$

Absolute Value Function

The rule of an absolute value function is:

$$f(x) = a|b(x - h)| + k \quad \text{vertex : (h, k)}$$

We can remove the “b” term (so $b = 1$) using properties of absolute values (which is especially useful when we are trying to find the rule of an absolute value function):

$$f(x) = a|(x - h)| + k$$

Ex: Write the following rule such that $b = 1$.

$$f(x) = 4|-9(x + 7)| - 6$$

$$f(x) = 4 \cdot |-9| \cdot |(x + 7)| - 6$$

$$f(x) = 4 \cdot 9|(x + 7)| - 6$$

$$f(x) = 36|(x + 7)| - 6$$

Absolute Value Function Basics:

- An absolute value function is a \vee or \wedge graphically, formed by 2 linear rays meeting at the vertex.
- The graph is symmetrical with an axis of symmetry at $x = h$.
- The slope of the rays is $\pm a$, provided $b = 1$. If $b \neq 1$, factor out the b term and remove it (as above) to determine the slope of the rays.
- If $-1 < a < 1$ and $a \neq 0$, the function is wider than the function when $a = 1$.
- If $-1 < a$ or $a > 1$, the function is narrower than when $a = 1$.
- Domain: $]-\infty, +\infty[$
- Range: $[k, +\infty[$ if $a > 0$; $]-\infty, k]$ if $a < 0$

Graphing an Absolute Value Function

There are 4 steps to graphing an absolute value function

- 1) Determine (and plot) the vertex (h, k)
- 2) Determine whether the function is upright: \vee ($a > 0$) or upside down: \wedge ($a < 0$)
- 3) Determine the slope of each ray ($\pm a$) (and use slope to find a point on each ray)
- 4) Connect the vertex and the points with two linear rays.

Ex: Graph the following absolute value functions

a) $f(x) = |x|$

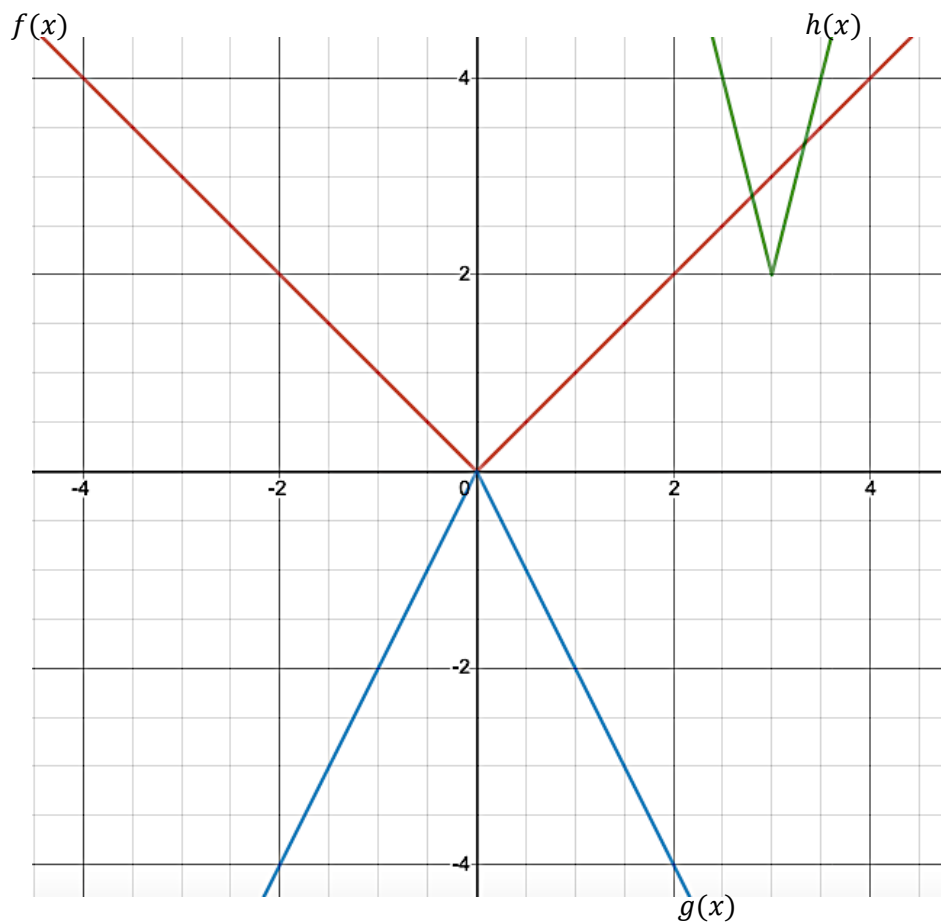
Vertex: $(0,0)$
Function is upright
Right ray slope: 1
Left ray slope: -1

b) $g(x) = -2|x|$

Vertex: $(0,0)$
Function is upside down
Right ray slope: -2
Left ray slope: 2

c) $h(x) = 2|-2(x - 3)| + 2$
 $h(x) = 2|-2|(x - 3)| + 2$
 $h(x) = 4|(x - 3)| + 2$

Vertex: $(3,2)$
Function is upright
Right ray slope: 4
Left ray slope: -4



Finding the Rule

Recall the rule of an absolute value function is $f(x) = a|b(x - h)| + k$, but the b value can be combined with the a value by using properties of absolute values. When finding the rule of an absolute value function, we use the simplified rule: $f(x) = a|(x - h)| + k$ (unless we are given the b value).

There are 4 basic cases (and variations of these) that might occur when asked to find the rule of an absolute value function:

- 1) Given the vertex and a point
- 2) Given the vertex (without it being identified as a vertex) and two points (with the same y -values)
- 3) Given multiple points on each ray
- 4) Given multiple points on one ray and one on the other ray, where 2 of the points share y -values

Case 1

If we are given the vertex and a point, we can find the rule of an absolute value function by:

- 1) Write the general form of the rule
- 2) Substitute in (h, k) given the vertex
- 3) Substitute in (x, y) given the additional point
- 4) Solve for a
- 5) Write the rule (replacing $a, h, and k$)

Ex: Determine the rule of an absolute value function has a vertex of $(3, 5)$ and passes through the point $(5, 8)$.

$$f(x) = a|(x - h)| + k$$

$$f(x) = a|(x - 3)| + 5$$

$$8 = a|(5 - 3)| + 5$$

$$3 = a|(2)|$$

$$3 = 2a$$

$$\frac{3}{2} = a$$

$$\therefore \text{the rule of the function is } f(x) = \frac{3}{2}|(x - 3)| + 5$$

Case 2

If we are given the vertex (without it being labeled as a vertex) and two points with the same y-value, we can find the rule of an absolute value function by:

- 1) Using the two points with the same y-value to determine h .
- 2) Check that the third point is the vertex (does $x = h$?)
- 3) Write the general form of the rule
- 4) Substitute in (h, k) given the vertex
- 5) Substitute in (x, y) given the additional point
- 6) Solve for a
- 7) Write the rule (replacing $a, h, \text{ and } k$)

Ex: Determine the rule of an absolute value function that passes through the points: $(4, 2)$, $(-2, 2)$, and $(1, -10)$.

Notice that the points $(4, 2)$ and $(-2, 2)$ have the same y-values. Because absolute value functions have an axis of symmetry at $x = h$, we know h must be half way between the two x values of these points.

$$h = \frac{x_1 + x_2}{2}$$

$$h = \frac{4 - 2}{2}$$

$$h = \frac{2}{2}$$

$$h = 1$$

$x = 1$ for the third point we are given, therefore, the point $(1, -10)$ must be the vertex of the function. So we can use that point and either other point to determine the rule

$$f(x) = a|(x - h)| + k$$

$$f(x) = a|(x - 1)| - 10$$

$$2 = a|(4 - 1)| - 10$$

$$12 = a|(3)|$$

$$12 = 3a$$

$$4 = a$$

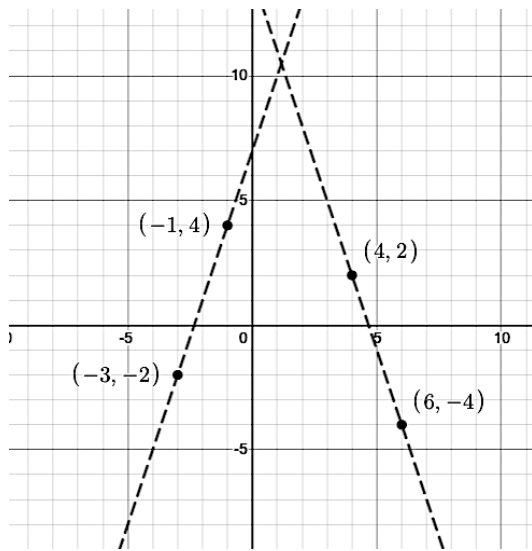
\therefore the rule of the function is $f(x) = 4|(x - 1)| - 10$

Case 3

If we are given at least two points on each ray, we can find the rule of an absolute value function by:

- 1) Make a sketch, plotting all 4 points and drawing two lines
- 2) Use two points on one of the lines to determine the rule of that line
- 3) Use the negative slope of the one found in step 2 and an additional point, to find the rule for the second line.
- 4) Find the intersection point of the two lines (this is the vertex)
- 5) Determine whether a is positive or negative (using the sketch)
- 6) Write the rule (replacing $a, h,$ and k)

Ex: Determine the rule of an absolute value function passing through the points $(-1, 4), (-3, -2), (4, 2),$ and $(6, -4)$.



Ray 1:

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

$$a = \frac{4 - (-2)}{-1 - (-3)}$$

$$a = \frac{6}{2}$$

$$a = 3$$

$$y = ax + b$$

$$y = 3x + b$$

$$4 = (3)(-1) + b$$

$$4 = -3 + b$$

$$7 = b$$

$$\therefore y = 3x + 7$$

Ray 2: $a = -3$

$$y = ax + b$$

$$y = -3x + b$$

$$2 = (-3)(4) + b$$

$$2 = -12 + b$$

$$14 = b$$

$$\therefore y = -3x + 14$$

Point of Intersection

$$3x + 7 = -3x + 14$$

$$6x + 7 = 14$$

$$6x = 7$$

$$x = \frac{7}{6} = h$$

$$y = 3x + 7$$

$$y = 3\left(\frac{7}{6}\right) + 7$$

$$y = \frac{7}{2} + 7$$

$$y = \frac{21}{2} = k$$

Write the Rule

Upside down, so a is negative

$$a = -3$$

$$\text{Vertex: } \left(\frac{7}{6}, \frac{21}{2}\right)$$

$$\therefore f(x) = -3\left|x - \frac{7}{6}\right| + \frac{21}{2}$$

Case 4

If we are given two points on one ray and one point on the other ray (that shares a y -value with one of the points on the other ray), we can:

1. Determine h
2. Determine a
3. Find the rule of one ray
4. Use h and the rule of one ray to solve for k
5. Write the rule (substituting in a , h , and k)

Ex: Determine the rule of an absolute value function that passes through the points $(-6, -7)$, $(-8, -11)$ on one ray, and $(0, -11)$ on the other ray.

Note that there are two points with a y -value of -11 .

Find h

$$h = \frac{x_1 + x_2}{2}$$

$$h = \frac{-8 + 0}{2}$$

$$h = -4$$

Determine slope for ray

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

$$a = \frac{-11 - (-7)}{-8 - (-6)}$$

$$a = \frac{-11 - (-7)}{-8 - (-6)}$$

$$a = \frac{-4}{-2}$$

$$a = 2$$

Determine sign of a

For the absolute value function,
 $a = -2$ because the function is
upside down.

Determine rule of ray

$$\begin{aligned}y &= ax + b \\-11 &= 2(-8) + b \\-11 &= -16 + b \\5 &= b\end{aligned}$$

Determine k

$$\begin{aligned}y &= 2x + 5 \\y &= 2(-4) + 5 \\y &= -8 + 5 \\y &= -3 \text{ so } k = -3\end{aligned}$$

Write rule

$$\begin{aligned}y &= a|x - h| + k \\y &= -2|x + 4| - 3\end{aligned}$$

$$\therefore y = -2|x + 4| - 3$$

Solving Absolute Value Functions

To solve for y , plug in x and solve (remember to the absolute value of a number is always positive).

To solve for x :

- 1) Isolate the absolute value
- 2) Create two cases, one where the $| \quad | = +$ and one where the $| \quad | = -$
- 3) Solve for x in each case
- 4) Check for extraneous answers (plug solution into equation)

Recall

$$\text{If: } |x| = 2$$

$$\text{Then } x = 2 \text{ and } x = -2$$

$$\text{So if: } |x + 6| = 2$$

$$\text{Then } x + 6 = 2 \text{ and } x + 6 = -2$$

$$\text{And } x = -4 \text{ and } x = -8$$

Ex: Solve $2|x + 4| - 2 = 0$

$$\begin{aligned} 2|x + 4| - 2 &= 0 \\ 2|x + 4| &= 2 \\ |x + 4| &= 1 \end{aligned}$$
$$\begin{array}{l} x + 4 = 1 \\ x = -3 \end{array} \quad \begin{array}{l} x + 4 = -1 \\ x = -5 \end{array}$$

Check $x = -3$

$$\begin{aligned} 2|x + 4| - 2 &= 0 \\ 2|-3 + 4| - 2 &= 0 \\ 2|1| - 2 &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0 \\ \text{True} \end{aligned}$$

Check $x = -5$

$$\begin{aligned} 2|x + 4| - 2 &= 0 \\ 2|-5 + 4| - 2 &= 0 \\ 2|-1| - 2 &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0 \\ \text{True} \end{aligned}$$

$$\therefore 2|x + 4| - 2 = 0 \text{ when } x = -3 \text{ and } x = -5$$

Ex: Solve $4|x + 2| + 6 = 1$


$$4|x + 2| + 6 = 1$$

$$4|x + 2| = -5$$

$$|x + 2| = -\frac{5}{4}$$

No solutions (because the absolute value of a term can never be negative).

Ex: Given $f(x) = |x - 2| - 1$ and $g(x) = -2x$, solve $f(x) = g(x)$

$$\begin{aligned} f(x) &= g(x) \\ |x - 2| - 1 &= -2x \\ |x - 2| &= -2x + 1 \end{aligned}$$

$$\begin{array}{ll} x - 2 = -2x + 1 & x - 2 = -(-2x + 1) \\ 3x = 3 & x - 2 = 2x - 1 \\ x = 1 & -x = 1 \\ & x = -1 \end{array}$$

Check $x = 1$

$$\begin{aligned} |1 - 2| - 1 &= -2(1) \\ |-1| - 1 &= -2 \\ 1 - 1 &= -2 \\ 0 &= -2 \end{aligned}$$

False (so not a solution)

Check $x = -1$

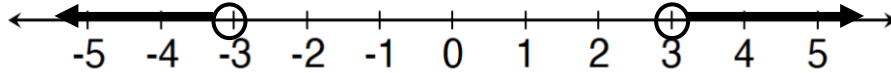
$$\begin{aligned} |-1 - 2| - 1 &= -2(-1) \\ |-3| - 1 &= 2 \\ 3 - 1 &= 2 \\ 2 &= 2 \end{aligned}$$

True

$\therefore f(x) = g(x)$
when $x = -1$

Solving Absolute Value Inequalities

Recall that $|x| > 3$ means that x (whatever is inside the absolute value) is more than 3 units away from 0. On a number line, this means:



So $|x| > 3$ means $x > 3$ and $x < -3$ (note the change of direction in the inequality)

We will use this when solving inequalities for absolute value functions.

To solve an absolute value inequality:

- 1) Isolate the absolute value
- 2) Create two cases (remember to flip the inequality when you take the negative in the second case)
- 3) Solve (remember to change the inequality if you multiply or divide by a negative number)
- 4) Check using equality
- 5) State solution

Ex: Solve $|x - 7| - 4 < -2$

$$\begin{array}{l}
 |x - 7| - 4 < -2 \\
 |x - 7| < 2 \\
 \swarrow \quad \searrow \\
 x - 7 < 2 \quad x - 7 > -2 \\
 x < 9 \quad \quad x > 5
 \end{array}$$

Check $x < 9$ (so use $x = 9$)

$$\begin{array}{l}
 |x - 7| - 4 = -2 \\
 |9 - 7| - 4 = -2 \\
 |2| - 4 = -2 \\
 2 - 4 = -2 \\
 -2 = -2 \\
 \text{True}
 \end{array}$$

Check $x > 5$ (so use $x = 5$)

$$\begin{array}{l}
 |x - 7| - 4 = -2 \\
 |5 - 7| - 4 = -2 \\
 |-2| - 4 = -2 \\
 2 - 4 = -2 \\
 -2 = -2 \\
 \text{True}
 \end{array}$$

So $|x - 7| - 4 < -2$ when x is greater than 5 and less than 9

\therefore the solution is $]5, 9[$

Ex: Solve $|2x| - 4 \geq 0$

$$\begin{array}{c}
 |2x| - 4 \geq 0 \\
 |2x| \geq 4 \\
 \swarrow \quad \searrow \\
 2x \geq 4 \quad 2x \leq -4 \\
 x \geq 2 \quad x \leq -2
 \end{array}$$

Check $x \geq 2$ (so use $x = 2$)

$$\begin{array}{l}
 |2x| - 4 = 0 \\
 |2 \cdot 2| - 4 = 0 \\
 |4| - 4 = 0 \\
 4 - 4 = 0 \\
 0 = 0 \\
 \text{True}
 \end{array}$$

Check $x \leq -2$ (so use $x = -2$)

$$\begin{array}{l}
 |2x| - 4 = 0 \\
 |2 \cdot -2| - 4 = 0 \\
 |-4| - 4 = 0 \\
 4 - 4 = 0 \\
 0 = 0 \\
 \text{True}
 \end{array}$$

So $|2x| - 4 \geq 0$ when x is greater than or equal to 2 and less than or equal to -2

\therefore the solution is $]-\infty, -2] \cup [2, +\infty[$

Ex: Given $f(x) = |x - 2|$ and $g(x) = 4x + 8$, determine the interval over which $f(x) \geq g(x)$

$$\begin{array}{c}
 f(x) \geq g(x) \\
 |x - 2| \geq 4x + 8 \\
 \swarrow \quad \searrow \\
 x - 2 \geq 4x + 8 \quad x - 2 \leq -(4x + 8) \\
 -3x \geq 10 \quad x - 2 \leq -4x - 8 \\
 x \leq -\frac{10}{3} \quad 5x \leq -6 \\
 \quad \quad \quad x \leq -\frac{6}{5}
 \end{array}$$

Check $x \leq -\frac{10}{3}$

$$\begin{array}{l}
 \left| -\frac{10}{3} - 2 \right| = 4 \left(-\frac{10}{3} \right) + 8 \\
 \left| -\frac{16}{3} \right| = -\frac{40}{3} + 8 \\
 \frac{16}{3} = -\frac{16}{3}
 \end{array}$$

False (so not a solution)

Check $x \leq -\frac{6}{5}$

$$\begin{array}{l}
 \left| -\frac{6}{5} - 2 \right| = 4 \left(-\frac{6}{5} \right) + 8 \\
 \left| -\frac{16}{5} \right| = -\frac{24}{5} + 8 \\
 \frac{16}{5} = \frac{16}{5}
 \end{array}$$

True

So $f(x) \geq g(x)$ when x is less than or equal to $-\frac{6}{5}$

\therefore the solution is $]-\infty, -\frac{6}{5}]$

Finding the Inverse of an Absolute Value Function

Important note: Remember that taking the inverse of a parabola gives us a sideways parabola that is not a function (and specifically not a square root function). When we take the inverse of an absolute value function, we will also get something that is not a function (and looks like a sideways absolute value function).

When we find the inverse of an absolute value function, we will get what look like two linear functions, and we must state the domain of the $f^{-1}(x)$, which was the range of $f(x)$.

Ex: Find the inverse of $f(x) = 2|x - 2| + 1$ and sketch both $f(x)$ and $f^{-1}(x)$.

$$f(x) = 2|x - 2| + 1 \quad \text{Domain: }]-\infty, +\infty[\quad \text{Range: } [1, +\infty[$$

Inverse

$$x = 2|y - 2| + 1$$

$$x - 1 = 2|y - 2|$$

$$\frac{x - 1}{2} = |y - 2|$$

$$\frac{x - 1}{2} = y - 2$$

$$\frac{x - 1}{2} + 2 = y$$

$$y = \frac{1}{2}x - \frac{1}{2} + 2$$

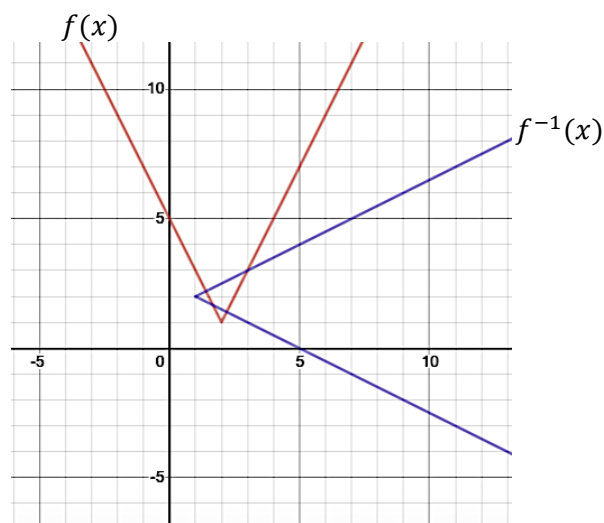
$$y = \frac{1}{2}x + \frac{3}{2}$$

$$-\left(\frac{x - 1}{2}\right) = y - 2$$

$$-\left(\frac{x - 1}{2}\right) + 2 = y$$

$$y = -\frac{1}{2}x + \frac{1}{2} + 2$$

$$y = -\frac{1}{2}x + \frac{5}{2}$$



\therefore the inverse of $f(x) = 2|x - 2| + 1$ is $y = \frac{1}{2}x + \frac{3}{2}$ and $y = -\frac{1}{2}x + \frac{5}{2}$ with a domain of $[1, +\infty[$