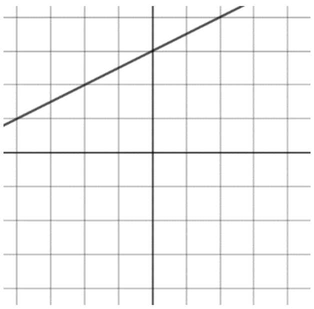
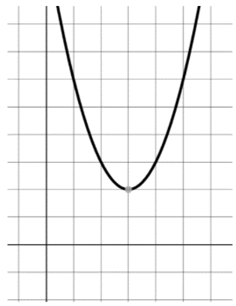
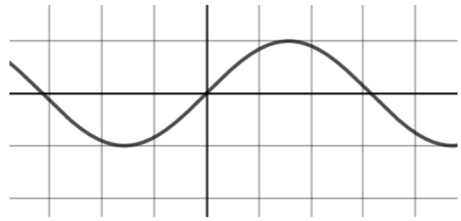
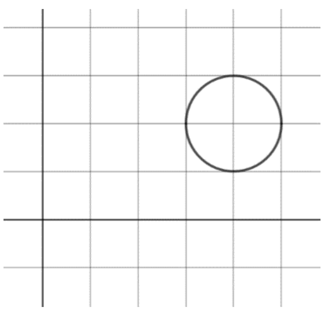
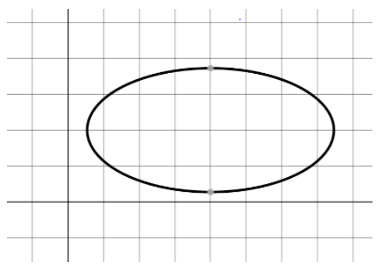
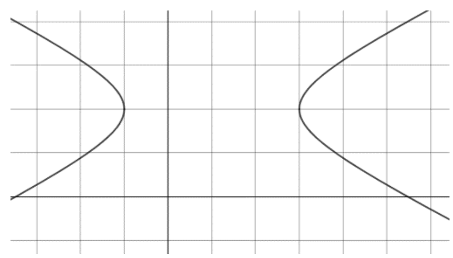
Properties of Functions

Functions: Every x in the domain of the function maps onto one y value. Must pass the vertical line test.

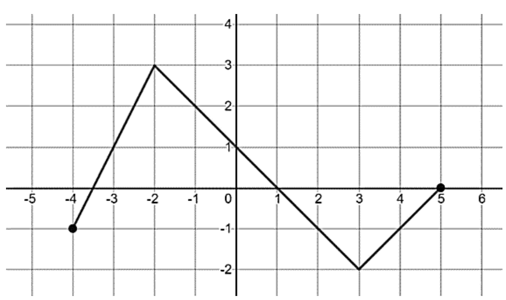
Examples:

Relations: An x can map onto more than one y value. Does not pass the vertical line test.

Recall the following definitions. Examples are based on the graph below.



Domain: the set of all possible x values.	Ex: Domain: $[-4, 5]$
Range: the set of all possible y values	Ex: Range: $[-2, 3]$
Initial value: the y-value at $x = 0$	Ex: Initial value $= 1$
Variation: the x values over which the function is increasing and decreasing	Ex: Increasing over $[-4, -2] \cup [3, 5]$ Decreasing over $]-2, 3]$
Extrema: any maximum and minimum points of the function	Ex: $(-2, 3) and\ (3, -2)$
Zeros: the x values at $y = 0$	Ex: Zeros are $x = -3.5$ and $x = 1$ and $x = 5$
Signs: the x values over which the function is above the x-axis (positive) and below the x-axis (negative)	Ex: Positive over $[-3.5, 1]$ Negative over $[-4, -3.5] \cup [1, 5]$

Below are some of the functions we'll be looking at in this course.

Function	Basic Form	Transformed
Absolute Value	$f(x) = \lvert x \rvert$	$f(x) = a\lvert b(x-h)\rvert + k$
Square Root	$f(x) = \sqrt{x}$	$f(x) = a\sqrt{b(x-h)} + k$
Rational	$f(x) = \frac{1}{x}$	$f(x) = \frac{a}{b(x-h)} + k$

The parameters a, b, h, and k, change the shape and location of the function

a:	Vertical stretch or compression and vertical reflection
b:	Vertical stretch or compression and horizontal reflection
h:	Horizonal shift
k:	Vertical shift

Inverse of a Function

Swapping the x and the y coordinates of the ordered pairs of any function or relation gives its inverse. If $f(x)$ is the function, $f^{-1}(x)$ is its inverse.

Ex: $R = \{(0,1),(2,3),(5,4)\}$ $\quad$ $R^{-1} = \{(1,0),(3,2),(4,5)\}$

We can think of the inverse of a function as a reflection over the line $y = x$.

The domain of a function becomes the range of its inverse.

The range of a function becomes the domain of its inverse.

The inverse of a function may be a relation or a function.

To solve for $f^{-1}(x)$, swap x and y and then isolate y.

Ex: Given $f(x) = 2x - 5$, find $f^{-1}(x)$.

$$y = 2x - 5$$

$$x = 2y - 5$$

$$x + 5 = 2y$$

$$\frac{x}{2} + \frac{5}{2} = y$$

$$f^{-1}(x) = \frac{x}{2} + \frac{5}{2}$$

For each function below, find $f^{-1}(x)$.

a) $\boldsymbol{f(x) = 2(x-6)^2 + 4}$

$$x = 2(y-6)^2 + 4$$
$$x - 4 = 2(y-6)^2$$
$$\frac{x}{2} - 2 = (y-6)^2$$
$$\sqrt{\frac{x}{2} - 2} = y - 6$$
$$\sqrt{\frac{x}{2} - 2} + 6 = y$$

$$f^{-1}(x) = \sqrt{\frac{x}{2} - 2} + 6$$

b) $\boldsymbol{f(x) = 2\sqrt{x+4} - 10}$

$$x = 2\sqrt{y+4} - 10$$
$$x + 10 = 2\sqrt{y+4}$$
$$\frac{x}{2} + 5 = \sqrt{y+4}$$
$$\left(\frac{x}{2} + 5\right)^2 = y + 4$$
$$\left(\frac{x}{2} + 5\right)^2 - 4 = y$$

$$f^{-1}(x) = \left(\frac{x}{2} + 5\right)^2 - 4$$

c) $\boldsymbol{f(x) = \frac{4}{3(x-2)} + 1}$

$$x = \frac{4}{3(y-2)} + 1$$
$$x - 1 = \frac{4}{3(y-2)}$$
$$(x-1)(3)(y-2) = 4$$
$$(3x-3)(y-2) = 4$$
$$y - 2 = \frac{4}{3x-3}$$
$$y = \frac{4}{3x-3} + 2$$

$$f^{-1}(x) = \frac{4}{3x-3} + 2$$

d) $\boldsymbol{f(x) = \frac{x+3}{x+4}}$

$$x = \frac{y+3}{y+4}$$
$$x(y+4) = y + 3$$
$$xy + 4x = y + 3$$
$$xy - y = -4x + 3$$
$$y(x-1) = -4x + 3$$
$$y = \frac{-4x+3}{x-1}$$

$$f^{-1}(x) = \frac{-4x+3}{x-1}$$

Ex: Given $f(x) = 4x + 6$, solve for $f^{-1}(14)$

$$f(x) = 4x + 6$$

$$x = 4y + 6$$

$$x - 6 = 4y$$

$$\frac{x-6}{4} = y$$

$$f^{-1}(x) = \frac{x-6}{4}$$

$$f^{-1}(x) = \frac{x-6}{4}$$

$$f^{-1}(14) = \frac{14-6}{4}$$

$$f^{-1}(14) = \frac{8}{4}$$

$$\boldsymbol{f^{-1}(14) = 2}$$

Ex: Given $f(x) = 6x - 2$, find $f^{-1}(x) = 3$

$$f(x) = 6x - 2$$

$$x = 6y - 2$$

$$x + 2 = 6y$$

$$\frac{x+2}{6} = y$$

$$f^{-1}(x) = \frac{x+2}{6}$$

$$f^{-1}(x) = \frac{x+2}{6}$$

$$\frac{x+2}{6} = 3$$

$$x + 2 = 18$$

$$x = 16$$

$$\therefore when\ f^{-1}(x) = 3, x = 16$$

Ex: Given $f(x) = \ -2(x+6)^2 - 3$, determine the domain of $f^{-1}(x)$

Remember that the domain of the inverse is the range of the function.

$f(x)$ is a parabola with a vertex at $(-6, -3)$and it opens down because the sign on *a* is negative.

So the range of $f(x)$ is $]-\infty, -3]$

Thus, the domain of $f^{-1}(x)$ is $]-\infty, -3]$

Operations and Composites

Operations (addition, subtraction, multiplication, division) can be performed on functions. For example, we can add two functions to create a new function.

Composites are when one function is applied to another function

Ex: $f\big(g(x)\big)\ or\ f \circ g$ "f of g of x"

Ex: $g\big(f(x)\big)\ or\ g \circ f$ "g of f of x"

Ex: Given $f(x) = 2x + 1$ and $g(x) = x^2 - 2x$, find $h(x)$ as defined below:

a) $h(x) = f(x) + g(x)$

$$h(x) = 2x + 1 + x^2 - 2x$$

$$h(x) = x^2 + 1$$

b) $h(x) = f(x) - g(x)$

$$h(x) = 2x + 1 - (x^2 - 2x)$$

$$h(x) = -x^2 + 4x + 1$$

c) $h(x) = f(x) \cdot g(x)$

$$h(x) = (2x + 1)\,(x^2 - 2x)$$

$$h(x) = 2x^3 - 4x^2 + x^2 - 2x$$

$$h(x) = 2x^3 - 3x^2 - 2x$$

d) $h(x) = \frac{f(x)}{g(x)}$

$$h(x) = \frac{(2x + 1)}{(x^2 - 2x)}$$

e) $h(x) = f(g(x))$

$2x + 1$ start with $f(x)$

$2(x^2 - 2x) + 1$ replace x with $g(x)$

$2x^2 - 4x + 1$ simplify

$$h(x) = 2x^2 - 4x + 1$$

f) $h(x) = g(f(x))$

$x^2 - 2x$ start with $g(x)$

$(2x + 1)^2 - 2(2x + 1)$ replace x with $f(x)$

$(2x + 1)(2x + 1) - 4x - 2$ simplify

$4x^2 + 2x + 2x + 1 - 4x - 2$

$4x^2 - 1$

$$h(x) = 4x^2 - 1$$

g) $h(-2) = f(g(-2))$

$2x + 1$ start with $f(x)$

$2(x^2 - 2x) + 1$ replace x with $g(x)$

$2((-2)^2 - 2(-2)) + 1$ replace x with -2

$2(4 + 4) + 1$ solve

17

$h(-2) = 17$

Alternate solution for g)

$g(x) = x^2 - 2x$ start with $g(x)$

$g(-2) = (-2)^2 - 2(-2)$ find $g(-2)$

$g(-2) = 8$

$f(x) = 2x + 1$ use $f(x)$

$f(8) = 2(8) + 1$ find $f(8)$

$f(8) = 17$

$h(-2) = 17$

h) $h(-2) = f^{-1}(g(-2)$ Find $f^{-1}(x)$ $f(x) = 2x + 1$ $x = 2y + 1$ $x - 1 = 2y$ $\frac{x-1}{2} = y$ $f^{-1}(x) = \frac{x-1}{2}$	$f^{-1}(x) = \frac{x-1}{2}$ $f^{-1}(g(x)) = \frac{x^2 - 2x - 1}{2}$ $f^{-1}(g(-2)) = \frac{(-2)^2 - 2(-2) - 1}{2}$ $f^{-1}(g(-2)) = \frac{4 + 4 - 1}{2}$ $f^{-1}(g(-2)) = \frac{7}{2}$

Ex: Given $f(x) = \frac{x+2}{x-1}$ and $g(x) = \frac{2x-3}{x-1}$ and $h(x) = 2x - 3$, what function results from $\frac{f}{g} \cdot h$?

$$\frac{f}{g} \cdot h = \frac{\frac{x+2}{x-1}}{\frac{2x-3}{x-1}} \cdot (2x - 3)$$

$$= \frac{x+2}{x-1} \cdot \frac{x-1}{2x-3} \cdot \frac{2x-3}{1}$$

$$= x + 2$$