# Vectors

May 9, 2022 1:55 PM

**Reminder:**

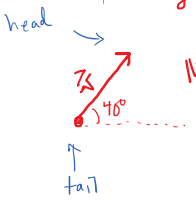
Scalar → quantity entirely defined by a number with no direction
ex. mass, time

Vector → quantity defined by a number and a direction.

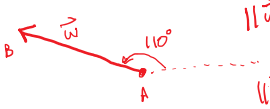
- written as: $\overrightarrow{AB}$, $\vec{v}$, $\vec{w}$
- magnitude or norm $\|\overrightarrow{AB}\|$ or $\|\vec{v}\|$
- direction usually given with respect to horizontal
- represent vectors geometrically or algebraically

**Geometrically**

- shown as an oriented segment with a tail and head whereby the length represents the norm.




$\|\vec{v}\| = 5\text{km}, 40°$



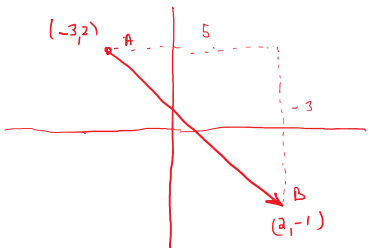

$\|\vec{w}\| = 40\frac{\text{km}}{\text{h}}, 110°$

$\|\overrightarrow{AB}\| = 40\frac{\text{km}}{\text{h}}, 110°$

**Algebraically**

- represented on a cartesian plane
- tail $(x_1, y_1)$, head $(x_2, y_2)$
- described as an ordered pair $(a, b)$ which represents the horizontal component $a$, where $a = x_2 - x_1$ and the vertical component $b$, where $b = y_2 - y_1$

NOT A POINT




$\overrightarrow{AB} = (a, b)$

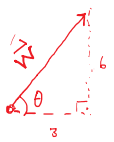
$\overrightarrow{AB} = (5, -3)$

x distance, y distance

$\|\overrightarrow{AB}\| = \sqrt{5^2 + 3^2}$

$\|\overrightarrow{AB}\| = \sqrt{34}$

- example. $\vec{w} = (3, 6)$. Find the magnitude and direction of $\vec{w}$




$c^2 = a^2 + b^2$

$= (3)^2 + (6)^2$

$c = \sqrt{45}$

$\|\vec{w}\| = \sqrt{45}$

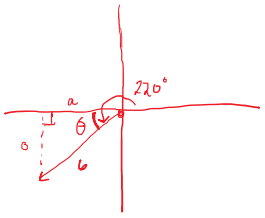
$\tan\theta = \left(\frac{6}{3}\right)$

$\tan^{-1}(2) = \theta$

$\theta = 63.43°$

$\|\vec{w}\| = \sqrt{45}, 63.43°$

- $\overrightarrow{AB}$ has a magnitude of 6 and a direction of 220°. Find the components.




$220 - 180 = 40°$

$\cos\theta = \frac{a}{6}$ $\quad \sin\theta = \frac{\text{opp}}{6}$

$\cos 40 = \frac{a}{6}$ $\quad \sin 40 = \frac{\text{opp}}{6}$

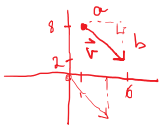
$a = 4.60$ $\quad \text{opp} = 3.86$

$(a, b)$

$(-4.60, -3.86)$

$\overrightarrow{AB} = (-4.60, -3.86)$

• $\vec{v}$ has a tail located at $(1,8)$ ($x_1, y_1$) and an arrowhead at $(6,2)$ ($x_2, y_2$). Find the magnitude of vector $\vec{v}$.




$a = x_2 - x_1$

$a = 6 - 1$

$a = 5$

$b = y_2 - y_1$

$= 2 - 8$

$= -6$

$c^2 = a^2 + b^2$

$c^2 = (5)^2 + (-6)^2$

$c^2 = 25 + 36$

$c^2 = 61$

$c = \sqrt{61}$

$\|\vec{v}\| = \sqrt{61}$

## RELATIONS BETWEEN TWO VECTORS

### ① Equivalent or Equipollent vectors

- same magnitude
- same direction
- same components

### ② Opposite vectors

- same magnitude
- opposite direction
- components have opposite signs

ex. $(a,b) = (-a,-b)$

$\vec{AB} = \vec{BA}$ OR $-\vec{AB}$

$\vec{v} = -\vec{v}$

### ③ Collinear

- parallel so the slope is the same
- components $ad - bc = 0$

ex. $\vec{u} = (3,5)$ ($a, b$) $\quad 3(10) - 5(6) = 0$

$\vec{v} = (6,10)$ ($c, d$) $\quad 0 = 0$ ✓

↑ on top of each other

### ④ Orthogonal Vectors

- perpendicular
- slopes are negative reciprocals
- components $ac + bd = 0$

ex. $\vec{u} = (1,-2)$ ($a, b$) $\quad (1)(2) + (-2)(1) = 0$

$\vec{v} = (2,1)$ ($c, d$) $\quad 0 = 0$ ✓

Orthogonal example

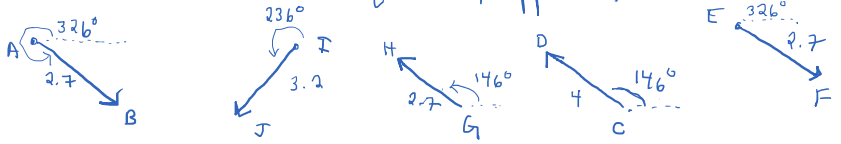
ex. Point A is $(1,2)$ and B is at $(5,6)$, representing vector $\vec{v}$.
Point C is $(6,-3)$ and D is at $(0,3)$, representing vector $\vec{u}$.
Show that $\vec{AB} \perp \vec{CD}$

slope of $\vec{AB}$: $\frac{6-2}{5-1} = \frac{4}{4} = 1$

slope of $\vec{CD}$: $\frac{0-6}{3+3} = \frac{-6}{6} = -1$

Since the two slopes are negative reciprocals, they're perpendicular vectors.

Vector Bank: Which are equivalent, opposite, collinear or orthogonal?




**Equivalent**

$\vec{AB}$ and $\vec{EF}$

**Opposite**

$\vec{AB}$ & $\vec{GH}$

$\vec{EF}$ & $\vec{GH}$

**Collinear**

$\vec{CD}$ & $\vec{GH}$

$\vec{AB}$ & $\vec{EF}$

**Orthogonal**

$\vec{IJ}$ & $\vec{AB}$, $\vec{CD}$, $\vec{GH}$, $\vec{EF}$

$\overline{AB}$ and $\overrightarrow{EF}$ | $\overline{AB}$ & $\overline{GH}$ | $\overrightarrow{EF}$ & $\overrightarrow{GH}$ | $\overline{CD}$ & $\overline{GH}$ | $\overrightarrow{AB}$ & $\overrightarrow{EF}$ | $\overline{IJ}$ & $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{GH}$, $\overrightarrow{EF}$

**Ex:** Equivalent, opposite, collinear, orthogonal?

- $\vec{w}$ (4, -6)
- $\vec{u}$ (-2, 3)
- $\vec{n}$ (-6, 12)
- $\vec{t}$ = (2, -8)
- $\vec{v}$ = (-4, 6)
- $\vec{s}$ = (-1, 4)
- $\vec{m}$ = (2, -3)
- $\vec{r}$ = (2, 1)

Equivalent : none

Opposite : $\vec{u}$ and $\vec{m}$; $\vec{w}$ and $\vec{v}$

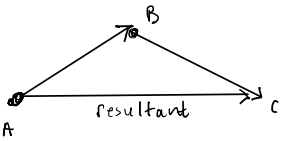
Collinear : $\vec{w}$, $\vec{v}$, $\vec{u}$ & $\vec{m}$

Orthogonal : $\vec{r}$ and $\vec{n}$

## Adding & Subtracting Vectors

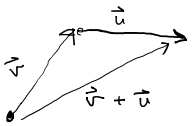
Vectors can be added or subtracted resulting in another vector.

Chasles Relation : given points A, B & C, then $\vec{AB} + \vec{BC} = \vec{AC}$

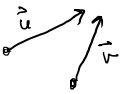



Geometric approach : sum of the vectors is determined by placing end to end ("tip to tail")

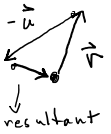
- resultant defined by tail of the first vector & head of the last




If subtracting, add the opposite vector




$\vec{v} - \vec{u}$ ?




$\vec{v} + -\vec{u} = \vec{v} - \vec{u}$

To determine the magnitude and direction of the resultant:

- measure it to scale
- SOH CAH TOA
- sine law $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- cosine law $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Algebraic approach: - sum of vectors determined by sum of the components

Given: $\vec{u} = (a, b)$, $\vec{v} = (c, d)$

$$\vec{u} + \vec{v} = (a+c, b+d)$$
$$\vec{u} - \vec{v} = (a-c, b-d)$$

*

ex. If $\vec{u} = (2, 3)$ and $\vec{v} = (-1, 5)$, then what is: $\vec{u} + \vec{v}$ ? $\vec{u} - \vec{v}$ ?

$\vec{u} + \vec{v} = (1, 8)$

$\vec{u} - \vec{v} = (3, -2)$

## Multiplying vectors by a scalar

- multiplying a vector by a scalar produces another vector

ex. $\vec{u} + \vec{u} + \vec{u} = 3\vec{u}$

(we call this scalar "k")

- if scalar k is positive, then the direction is the same
- if scalar k is negative, then the direction is the opposite

$$\| k\vec{u} \| = |k| \cdot \|\vec{u}\|$$

- given scalar k and vector $\vec{v} = (a, b)$, then $k\vec{v} = k(a, b) = (ka, kb)$

ex. vector $\vec{v} = (1, 4)$ then $-5\vec{v} = (-5, -20)$

Properties

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$
$$(k_1 k_2)\vec{u} = (k_2 k_1)\vec{u}$$
$$(k_1 + k_2)\vec{u} = k_1\vec{u} + k_2\vec{u}$$

example: Simplify: a) $-2(3\vec{u}) + 4\vec{u} = -2\vec{u}$

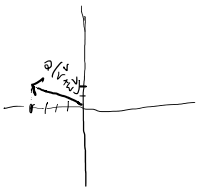
b) $2\vec{v} - 5\vec{u} + 3(\vec{v} + \vec{u}) - (2\vec{v} + \vec{u}) = 3\vec{v} - 3\vec{u}$

c) Given $\vec{u} = (1, 2)$ $\vec{v} = (1, 0)$ $\vec{w} = (-3, 1)$, draw

i) $2(\vec{v} + \vec{w})$.

$\downarrow$

$2(-2, 1) = (-4, 2)$



ii) $3\vec{w} - 2\vec{u}$

$3\vec{w} = (-9, 3)$

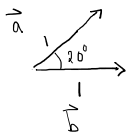
$2\vec{u} = (2, 4)$

$3\vec{w} - 2\vec{u} = (-11, -1)$

Note: a (unit vector) has a magnitude of 1

Ex. $\vec{a}$ and $\vec{b}$ are unit vectors that have an angle of 20° between them. Determine the magnitude and direction of $\|3\vec{a} - 2\vec{b}\|$.

Unit vectors:




Components of $\vec{a}$: x: $1 \cos 20° = 0.940$ $a \rightarrow 3(0.940) = 2.82$

y: $1 \sin 20° = 0.342$ $b \rightarrow 3(0.342) = 1.026$

Components of $\vec{b}$: x: 1 $c \rightarrow 2(1) = 2$

y: 0 $d \rightarrow 2(0) = 0$

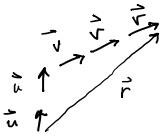
Vector subtraction: $(a - c, b - d)$

$(0.82, 1.026)$

## Linear Combination

- since a vector can be written as the sum of two other vectors, which themselves can be expressed as a product of a vector and a scalar, then a vector can be expressed as a <u>linear combination</u> of the other vectors.

$\vec{w} = k_1\vec{u} + k_2\vec{v}$

ex. 

$\vec{r} = 2\vec{u} + 3\vec{v}$

To solve for the scalar quantities, we use a system of equations with the components.

ex. $\vec{w} = \langle 16, 26\rangle$ can be expressed as a linear combination of $\vec{v} = \langle 3, 4\rangle$ and $\vec{u} = \langle 2, 5\rangle$

$\vec{w} = k_1\vec{u} + k_2\vec{v}$ ← what values of $k_1$ and $k_2$ result in vector $\vec{w}$?

$(16, 26) = k_1(2,5) + k_2(3,4)$

x y components: x y x y

add on the way down →

$x$: $(16 = 2k_1 + 3k_2)\ 5$

$+$ $y$: $(26 = 5k_1 + 4k_2)\ -2$ ← one sign needs to be negative if you're adding. Stay positive if subtracting

$80 = 10k_1 + 15k_2$

$+\ -52 = -10k_1 + (-8)k_2$

$28 = 7k_2$

$4 = k_2$

$16 = 2k_1 + 3(4)$

$16 - 12 = 2k_1$

$4 = 2k_1$

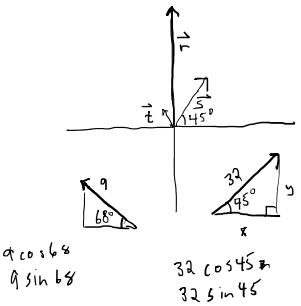
$2 = k_1$

$\vec{w} = k_1\vec{u} + k_2\vec{v}$

$\vec{w} = 2\vec{u} + 4\vec{v}$

ex. Express $\vec{r}$ as a linear combination of $\vec{s}$ and $\vec{t}$

$\|\vec{s}\| = 32\,[45°]$, $\|\vec{r}\| = 208\,[90°]$, $\|\vec{t}\| = 9\,[112°]$




$9\cos 68$
$9\sin 68$

$32\cos 45$
$32\sin 45$

$\vec{r} = \langle 0, 208\rangle$

$\vec{s} = \langle 22.63, 22.63\rangle$

$\vec{t} = \langle -3.37, 8.34\rangle$

$\vec{r} = k_1\vec{s} + k_2\vec{t}$

$-$ $x$: $(0 = 22.63k_1 - 3.37k_2)$ subtract!

$y$: $(208 = 22.63k_1 + 8.34k_2)$

$-208 = -11.71k_2$

$17.76 = k_2$

$0 = 22.63k_1 - 3.37(17.76)$

$0 = 22.63k_1 - 59.8699$

$2.65 = k_1$

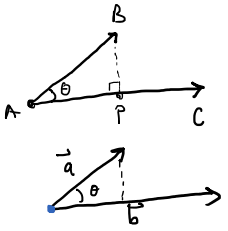
$\vec{r} = 2.65\vec{s} + 17.76\vec{t}$

# Scalar Product AKA Dot Product

Two vectors can be multiplied. The result is a scalar that corresponds to the sum of the products of their components:

$\vec{u} \cdot \vec{v} = ac + bd$, given $\vec{u} = (a,b)$ and $\vec{v} = (c,d)$

It also corresponds to the product of the directed length of the orthogonal projection of one vector onto the other, and the magnitude of the same vector.

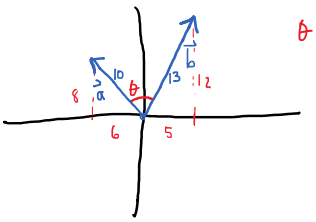



$\vec{AB} \cdot \vec{AC} = \vec{AP} \cdot \vec{AC}$

$a \cdot b = \|a\| \cdot \|b\| \cdot \cos\theta$

Where $\|a\|$ = magnitude of a
$\|b\|$ = magnitude of b
$\theta$ = must be enclosed bw a & b

Ex. Calculate the dot product of vectors $\vec{a}$ and $\vec{b}$




$\theta = 59.5°$

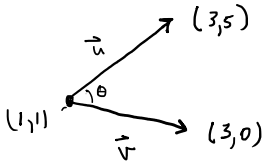
Method 2

$a \cdot b = 10 \cdot 13 \cdot \cos(59.5°)$
$= 65.98$
$= 66$ rounded

Method 1

$a \cdot b = (-6 \cdot 5) + (8 \times 12)$
$= -30 + 96$
$= 66$ (rounded)

Ex.



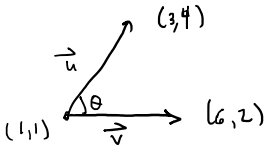

solve for θ

$\vec{u} \cdot \vec{v} = ac + bd$
$\vec{u} \cdot \vec{v} = (2)(2) + (4)(-1)$
$\vec{u} \cdot \vec{v} = 0$
$\therefore \theta = 90°$

if $\theta = 90°$, then $\|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos 90°$
$= 0$

if $\theta = 0°$ (collinear or equivalent)
then $\|\vec{u}\| \cdot \|\vec{v}\| \cdot (1)$

Ex. Solve for θ.




$\vec{u} = (2,3)$
$\vec{v} = (5,1)$
$\|\vec{u}\| = \sqrt{2^2+3^2} = \sqrt{13}$
$\|\vec{v}\| = \sqrt{26}$

$\vec{u} \cdot \vec{v} = ac + bd$
$= 2(5) + 3(1)$
$\vec{u} \cdot \vec{v} = 13$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

$13 = \sqrt{13} \cdot \sqrt{26} \cdot \cos\theta$

$\frac{\sqrt{2}}{2} = \cos\theta$

$\theta = 45^\circ$