Conics

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Locus / Loci (plural)

- a set of points that have a common metric property
  ex. a circle is a locus of points all equal distance from the centre

Conic

- the figure formed by the intersection of a plane with a conical surface.
  - circle, ellipse, hyperbola, parabola
  - always a locus of points

CIRCLE

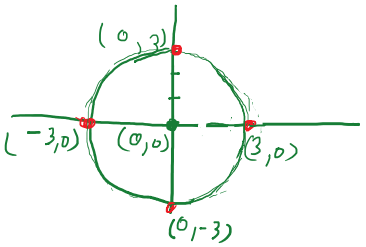
- See above for locus (equidistant from the centre)

It's a relation, not a function

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Basic equation, centred at the origin:

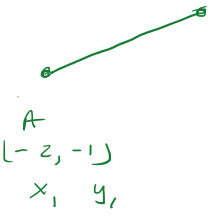
$$x^2 + y^2 = r^2$$



$$(x-0)^2 + (y-0)^2 = (3)^2$$

$$x^2 + y^2 = 9$$

positive & negative roots when solving!



Transformed equation: $(x-h)^2 + (y-k)^2 = r^2$

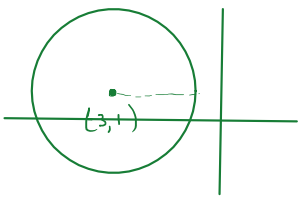
centre of the circle (h, k)

Graph the following:

$$(x+3)^2 + (y-1)^2 = 8$$

r^2

$r = 2.8$



$(-3, 1)$
h, k

Ex. $(2, -1)$ is on the locus of a circle centred at $(-1, -2)$. What's the equation of the circle?

$$r^2 = (x-h)^2 + (y-k)^2$$

$$r^2 = (2+1)^2 + (-1+2)^2$$

$r = \pm\sqrt{10}$

$r^2 = (x-h)^2 + (y-k)^2$

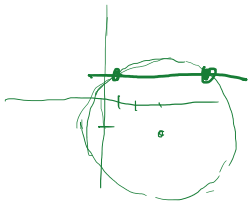
$r^2 = (x+1)^2 + (y+2)^2$

$10 = (x+1)^2 + (y+2)^2$

$r^2 = (2+1)^2 + (-1+2)^2$

$r = \pm\sqrt{10} \quad \approx \pm 3.1622$

Ex. $(x-3)^2 + (y+1)^2 = 16$ Solve for x when $y = 2$.



$(x-3)^2 + (2+1)^2 = 16$

$(x-3)^2 + 9 = 16$

$(x-3)^2 = 7$

$x - 3 = \pm\sqrt{7}$

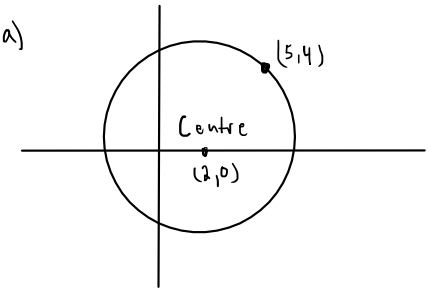
$x = \sqrt{7} + 3$ and $-\sqrt{7} + 3$

rewritten → $3 + \sqrt{7}$ and $3 - \sqrt{7}$

Finding the rule given the standard form and vice versa (converting)

$r^2 = (x-h)^2 + (y-k)^2$ → General form

a)



$r^2 = (x-2)^2 + y^2$

$r^2 = (5-2)^2 + (4)^2$

$r^2 = (3)^2 + (4)^2$

$= 9 + 16$

$= 25$

Standard form: ✓ $(x-2)^2 + y^2 = 25$

General form: ✓ $x^2 + y^2 - 4x - 21 = 0$

General form

$x^2 + y^2 + ax + by + c = 0$

$x^2 - 4x + 4 + y^2 = 25$

✓ $x^2 + y^2 - 4x - 21 = 0$

Why is this a circle?

Coefficients for x^2 and y^2 are the same.

b) $x^2 + 6x + y^2 - 2y + 1 = 0$ ← write in standard form

$(x^2 + 6x + 9) + (y^2 - 2y + 1) = -1 + 9 + 1$

$(x+3)^2 + (y-1)^2 = 9$

c) $x^2 + y^2 - 8x + 4y = -4$

$(x^2 - 8x + 16) + (y^2 + 4y + 4) = -4 + 16 + 4$

$(x-4)^2 + (y+2)^2 = 16$

Practice 1. $x^2 + y^2 - 2x + 6y + 3 = 0$

$x^2 - 2x + 1 + y^2 + 6y + 9 = -3 + 1 + 9$

$(x-1)^2 + (y+3)^2 = 7$

2. $x^2 + y^2 + 4x - 8y - 5 = 0$

$(x^2 + 4x + 4) + (y^2 - 8y + 16) = 0 + 4 + 16$ (+5)

$(x+2)^2 + (y-4)^2 = 20 + 5$

(25)

Quadratic

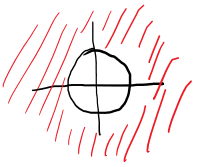
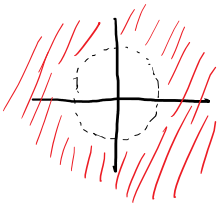
$x^2 + 6x + 21$

$x^2 + 6x + 9 = -21 + 9$

$(x+3)^2 = -12$

$y = (x+3)^2 + 12$

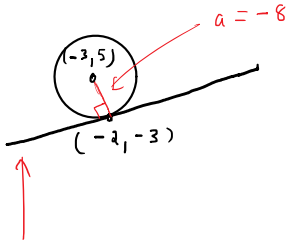
Inequalities

- if $(x-h)^2 + (y-k)^2 \leq r^2$ solid line, shade inside
- if $(x-h)^2 + (y-k)^2 < r^2$ dotted line, shade inside
- if $(x-h)^2 + (y-k)^2 \geq r^2$  solid line, shade outside
- if $(x-h)^2 + (y-k)^2 > r^2$  dotted line, shade outside

Tangent Line

- any line tangent to a circle touches only once and it's perpendicular to the radius
- Reminder: perpendicular lines have slopes that are negative reciprocals (flip the fraction & change the sign)

ex. What is the equation of the tangent line below?



$y = \frac{1}{8}x + b$

Slope of radius: $a = \frac{\Delta y}{\Delta x}$

$a = \frac{-3-5}{-2+3} = -\frac{8}{1} = -8$

∴ Slope of the line segment:

$-8 \rightarrow +\frac{1}{8}$

$y = \frac{1}{8}x + b$

$-3 = \frac{1}{8}(-2) + b$

$-3 = -\frac{1}{4} + b$

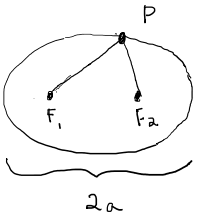
$-\frac{11}{4} = b$

$-8 \rightarrow +\frac{1}{8}$

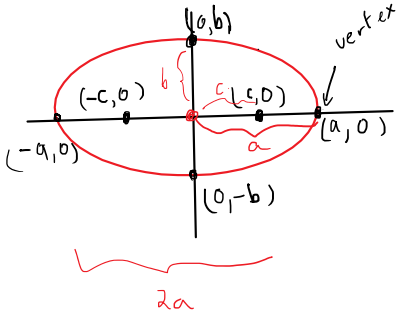
$\rightarrow$ Rule: $y = \frac{1}{8}x - \frac{11}{4}$

Ellipses

Definition: a set of points in a plane, the sum of whose distances from two fixed points (called foci) is a constant.



Focal radii property: $PF_1 + PF_2 = 2a$
(entire length of the major axis)



Half lengths!

a = always from the centre to the vertices of the major axis

b = centre to vertices of the minor axis

c = centre to foci

- Basic rule: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad a > b > 0$

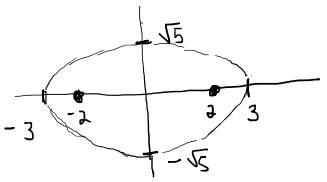
has foci at $(\pm c, 0)$ where

$c^2 = a^2 - b^2$
(larger, smaller)

Example: Graph: $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$a^2 = 9$, $\therefore a = \pm 3$

$b^2 = 5$, $\therefore b = \pm\sqrt{5}$



$c^2 = 9 - 5$

$c^2 = 4$

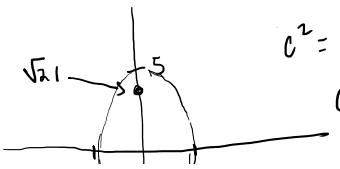
$c = \pm 2$

Graph: $\frac{x^2}{4} + \frac{y^2}{25} = 1$

$c^2 = a^2 - b^2$ $\rightarrow$ * just whichever is larger of a^2 or b^2 first!

$c^2 = 25 - 4$

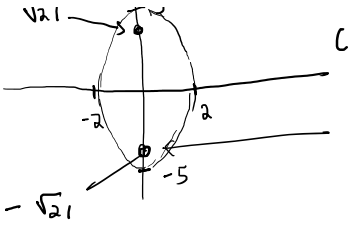
$c = \pm\sqrt{21}$



(4) 25

$a^2 = 4$, $a = \pm 2$.

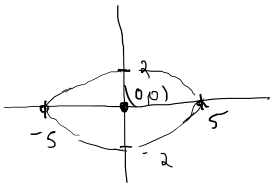
$b^2 = 25$, $b = \pm 5$



$c = \pm\sqrt{21}$ first!

Foci always on the MAJOR AXIS

Graph, identify the centre:



not equal or it would be a circle

$4x^2 + 25y^2 = 100$ ← must = 1

100 100 100

$\frac{x^2}{25} + \frac{y^2}{4} = 1$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Standard form when transformed, is as expected:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$c^2 = a^2 - b^2$

$c^2 = 36 - 20$

$c^2 = 16$

$c = \pm 4$

ex. $\frac{(x-4)^2}{20} + \frac{(y+2)^2}{36} = 1$

$a^2 = 20$

$a = \pm\sqrt{20} = \pm 2\sqrt{5}$

$b^2 = 36$

$b = \pm 6$

Graph?

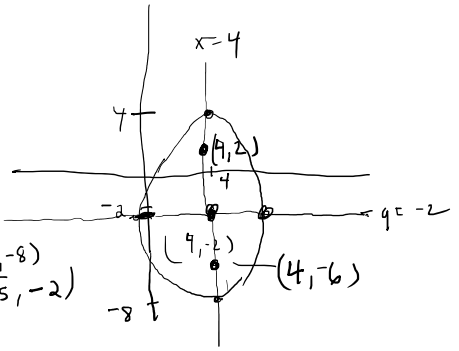
Centre? $(4, -2)$

Foci? $(4,2)$, $(4,-6)$

Vertices?

maj axis: $(4,4)$, $(4,-8)$

minor axis: $(4 \pm 2\sqrt{5}, -2)$



General form → Standard form

- Same as circle (group terms & move integer over)
- complete the square

$x^2 + 2y^2 - 6x + 4y + 7 = 0$

$(x^2 - 6x \quad) + (2y^2 + 4y \quad) = -7$

$(x^2 - 6x + 9) + 2(y^2 + 2y + 1) = -7 + 9$

$= -7 + 9 + 2$

$(x-3)^2 + 2(y+1)^2 = 4$

4 4 4

$(x-3)^2 + (y+1)^2 = 1$

Centre: $(3, -1)$

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{a} = 1$$

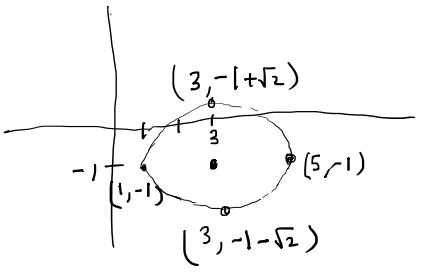
major axis

minor axis

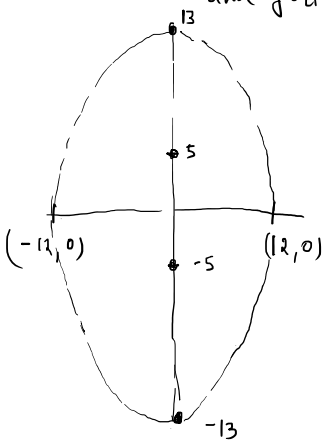
$a = \pm 2$

$b = \pm\sqrt{2}$

Centre: $(3, -1)$



Example: Write the equation of the ellipse with x intercepts at ± 12, and foci located at $(0,5)$ and $(0,-5)$, centred at the origin.



$c^2 = a^2 - b^2$

$(5)^2 = a^2 - (12)^2$

$25 = a^2 - 144$

$a^2 = 169$

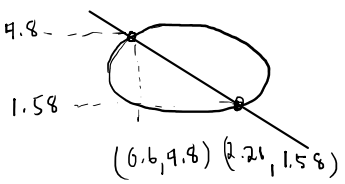
$a = \pm 13$

$$\frac{x^2}{144} + \frac{y^2}{169} = 1$$

Points of intersection

- you can use any method for solving for the coordinate(s) of the point(s) of intersection but substitution is the most seamless

Ex. Solve for the points of intersection between $\frac{(x+2)^2}{36} + \frac{(y-3)^2}{4} = 1$ and $y = -2x + 6$



$$\frac{(x+2)^2}{36} + \frac{9(y-3)^2}{9 \cdot 4} = 1$$

$$\frac{(x+2)^2 + 9(y-3)^2}{36} = 1$$

$(x+2)^2 + 9(y-3)^2 = 36$

Substitute in the linear function →

$(x+2)^2 + 9(-2x+6-3)^2 = 36$

$(x+2)^2 + 9(-2x+3)^2 = 36$

$x^2 + 4x + 4 + 9(-2x+3)(-2x+3) = 36$

$x^2 + 4x + 4 + (-18x+27)(-2x+3) = 36$

$4 \times 3 \times 2$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{104 \pm \sqrt{(104)^2 - 4(37)(49)}}{2(37)}$$

$$\frac{104 \pm 18\sqrt{11}}{74}$$

$$x^2 + 4x + 4 + (36x^2 - 54x - 54x + 81) = 36$$

$$x^2 + 4x + 4 + 36x^2 - 108x + 81 = 36$$

$$37x^2 - 104x + 49 = 0$$

$x = 2.21$ $\quad y = -2(2.21) + 6 = 1.58$

$x = 0.60$ $\quad y = -2(0.60) + 6 = 4.8$