Square Root Functions

Properties of radicals

Property	Example
$\sqrt[n]{a^m} = a^{m/n}$	$\sqrt[5]{a^2} = a^{2/5}$
$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$	$\sqrt{7} \cdot \sqrt{5} = \sqrt{35}$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$	$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$
$m\sqrt{a} \cdot n\sqrt{b} = mn\sqrt{ab}$	$2\sqrt{3} \cdot 4\sqrt{5} = 8\sqrt{15}$
$\frac{m\sqrt{ab}}{n\sqrt{b}} = \frac{m\sqrt{a}}{n}$	$\frac{5\sqrt{3 \cdot 4}}{2\sqrt{4}} = \frac{5\sqrt{3}}{2}$

Breaking apart radicals

Ex: $\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12} = 2\sqrt{4 \cdot 3} = 2\sqrt{4} \cdot \sqrt{3} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

Ex: $\frac{\sqrt{60}}{\sqrt{3}} = \sqrt{\frac{60}{3}} = \sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$

Rationalizing (getting rid of a radical in the denominator of a fraction)

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

$$\frac{a}{\sqrt{b} + \sqrt{c}} = \frac{a}{\sqrt{b} + \sqrt{c}} \cdot \frac{\sqrt{b} - \sqrt{c}}{\sqrt{b} - \sqrt{c}} = \frac{a\sqrt{b} - a\sqrt{c}}{b - c}$$

FOIL

$$(\sqrt{b} + \sqrt{c})(\sqrt{b} - \sqrt{c})$$

$$b - \sqrt{b}\sqrt{c} + \sqrt{b}\sqrt{c} - c$$

$$b - c$$

Square Root Function

A square root function is the inverse of a quadratic.

It is a semi-parabolic curve.

Rule

$$y = a\sqrt{b(x-h)} + k \qquad \text{vertex : (h. k)}$$

or we can remove the "b" term using properties of radicals and have

$$y = a\sqrt{\pm(x-h)} + k$$

Ex: Remove the "b" term from the rule: $y = 2\sqrt{-9x+27} + 1$

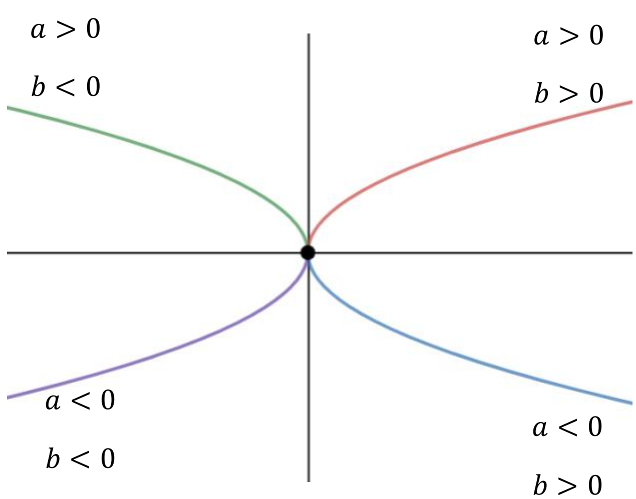
$$y = 2\sqrt{-9(x-3)} + 1$$

$$y = 2\sqrt{9}\sqrt{-(x-3)} + 1$$

$$y = 2 \cdot 3\sqrt{-(x-3)} + 1$$

$$y = 6\sqrt{-(x-3)} + 1$$

There are 4 possibilities for the direction and variation of square root functions.



If:

- $a < 0$, then range is $]-\infty, k]$
- $a > 0$, then range is $[k, +\infty[$
- $b < 0$, then domain is $]-\infty, h]$
- $b > 0$, then domain is $[h, +\infty[$

Sketches are useful to give us a general idea of the vertex of a function and its direction.

Sketch the following functions:

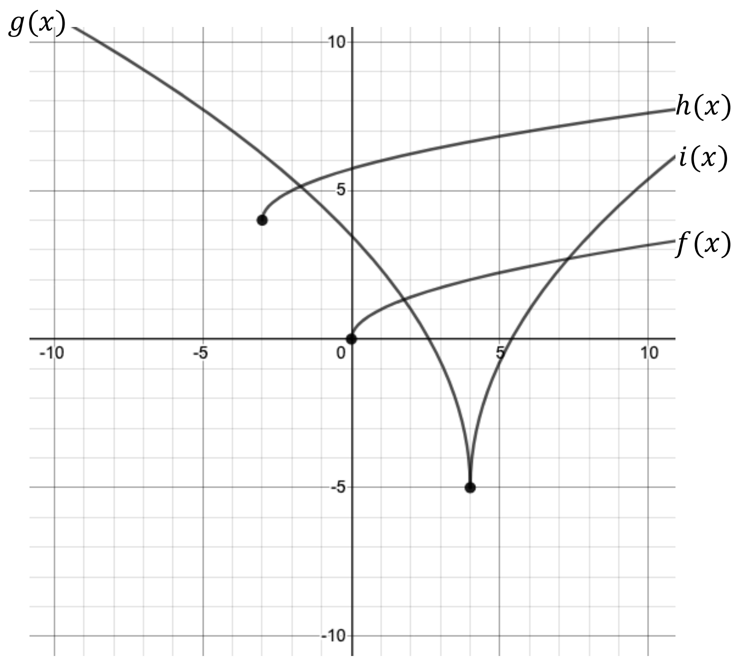
$f(x) = \sqrt{x}$ vertex: $(0, 0)$ $a > 0$ $b > 0$

$g(x) = 3\sqrt{-2(x-4)} - 5$ vertex: $(4, -5)$ $a > 0$ $b < 0$

$h(x) = \sqrt{x+3} + 4$ vertex: $(-3, 4)$ $a > 0$ $b > 0$

$i(x) = 3\sqrt{2x-8} - 5$

$= 3\sqrt{2(x-4)} - 5$ vertex: $(4, -5)$ $a > 0$ $b > 0$



Finding the Rule

There are 3 steps to finding the rule of a square root function.

1) Decide which rule to use:

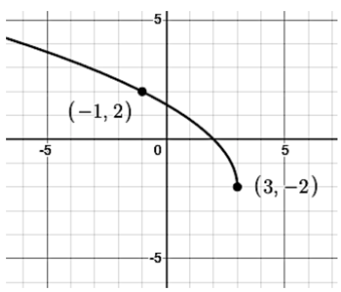
$$y = a\sqrt{(x-h)} + k \ or \ y = a\sqrt{-(x-h)} + k$$

In choosing the rule, sketch the function to determine its direction.

2) Plug in the vertex for h and k
3) Plug in a given point (x, y) and solve for a

Ex: Find the rule of a square root function with a vertex at $(3, -2)$ and passing through the point $(-1, 2)$

Sketch the function to determine the sign of b.



Given the sketch, we know $a > 0 \ and \ b < 0$

So use the rule $y = a\sqrt{-(x-h)} + k$

Plug in h and k: $y = a\sqrt{-(x-3)} - 2$

Plug in x and y: $2 = a\sqrt{-(-1-3)} - 2$

Solve for a $4 = a\sqrt{4}$

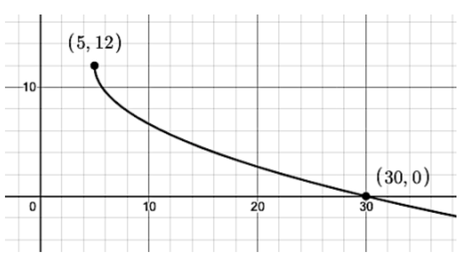
$$4 = 2a$$

$$2 = a$$

$$\therefore y = 2\sqrt{-(x-3)} - 2$$

Ex: Find the rule of a square root function with a vertex at $(5, 12)$ and passing through the point $(30, 0)$

Sketch the function to determine the sign of b.



Given the sketch, we know $a < 0 \ and \ b > 0$

So use the rule $y = a\sqrt{(x-h)} + k$

Plug in h and k: $y = a\sqrt{(x-5)} + 12$

Plug in x and y: $0 = a\sqrt{(30-5)} + 12$

Solve for a $-12 = a\sqrt{25}$

$$-12 = 5a$$

$$-\frac{12}{5} = a$$

$$\therefore y = -\frac{12}{5}\sqrt{(x-5)} + 12$$

Graphing a Square Root Function

There are 5 steps to graphing a square root function, given the rule.

1) Determine the vertex
2) Determine the signs of a and b (and thus the direction of the function)
3) Create a table of values for additional points. (Pick a few x values higher than h if b is positive and lower than h if b is negative.)
4) Solve for the corresponding y values
5) Plot points and connect the dots

Ex: Graph the function $y = -2\sqrt{-3x + 12} - 2$

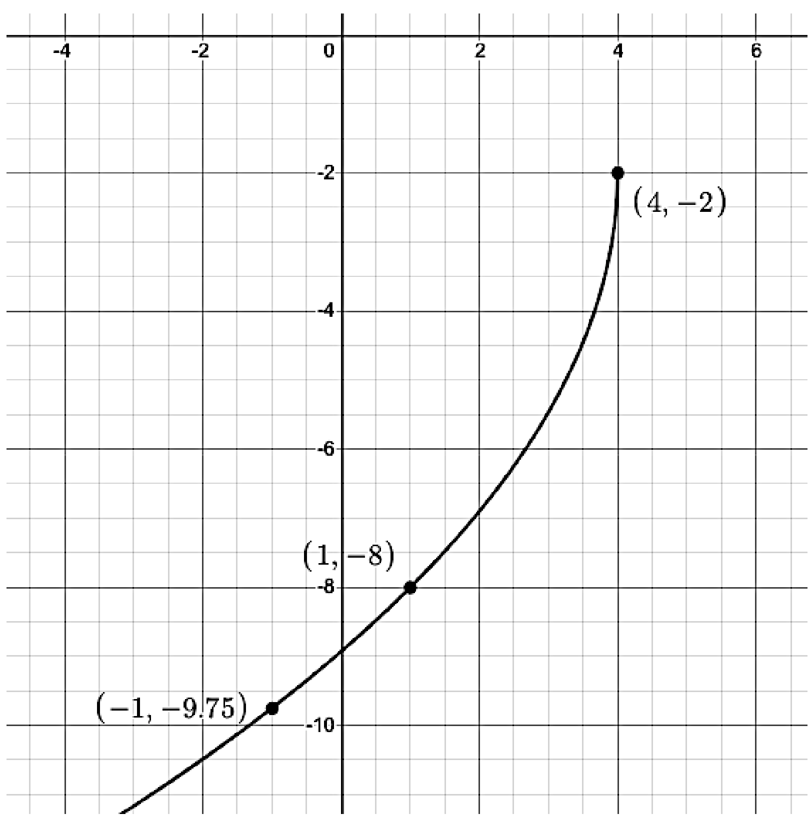
$y = -2\sqrt{-3x + 12} - 2$

$y = -2\sqrt{-3(x - 4)} - 2$ Therefore, vertex: $(4, -2)$ $a < 0$ $b < 0$

x	y
4	-2
1	-8
-1	-9.75

$$y = -2\sqrt{-3(x - 4)} - 2$$
$$y = -2\sqrt{-3(1 - 4)} - 2$$
$$y = -2\sqrt{9} - 2$$
$$y = -8$$

$$y = -2\sqrt{-3(x - 4)} - 2$$
$$y = -2\sqrt{-3(-1 - 4)} - 2$$
$$y = -2\sqrt{15} - 2$$
$$y = -9.75$$



Solving Square Root Functions

To solve for y, plug in x and solve (remember to only take the positive root).

To solve for x:

1) Isolate the radical
2) State domain restrictions (the value under the radical cannot be negative)
3) Solve for x
4) Check for extraneous answers (plug solution into equation)

Ex: Given $y = 2\sqrt{2x-4}$, solve for y when $x = 4$

$$y = 2\sqrt{2x-4}$$
$$y = 2\sqrt{2(4)-4}$$
$$y = 2\sqrt{4}$$
$$y = 4$$

Ex: Given $y = 2\sqrt{2x-4}$, solve for x when $y = 0$

$$y = 2\sqrt{2x-4}$$
$$0 = 2\sqrt{2x-4}$$
$$0 = \sqrt{2x-4}$$
$$0 = 2x-4$$
$$4 = 2x$$
$$2 = x$$

Solution: $x = 2$

Domain Restrictions

$$2x - 4 \geq 0$$
$$2x \geq 4$$
$$x \geq 2$$

Check for extraneous answers

$$y = 2\sqrt{2x-4}$$
$$0 = 2\sqrt{2(2)-4}$$
$$0 = 2\sqrt{0}$$
$$0 = 0$$

True

Ex: Given $y = 2\sqrt{x-3}$, solve for x when $y = 4$

$$4 = 2\sqrt{x-3}$$
$$2 = \sqrt{x-3}$$
$$4 = x-3$$
$$7 = x$$

Solution: $x = 7$

Domain Restrictions

$$x - 3 \geq 0$$
$$x \geq 3$$

Check for extraneous answers

$$4 = 2\sqrt{7-3}$$
$$4 = 2\sqrt{4}$$
$$4 = 4$$

True

Ex: Given $y = 2\sqrt{x+3} + 2$, solve for the zero of the function

$$0 = 2\sqrt{x+3} + 2$$
$$-2 = 2\sqrt{x+3}$$
$$-1 = \sqrt{x+3}$$
$$No\ Solution$$

> Note: There is no solution because when working with square root functions, we always take the positive root. Anytime we have $\sqrt{\ } = -$, there will be no solution

Ex: Determine when the following two functions intersect

$f(x) = 2\sqrt{x+4} - 1$ and $g(x) = x$

$$x = 2\sqrt{x+4} - 1$$
$$x + 1 = 2\sqrt{x+4}$$
$$\frac{x+1}{2} = \sqrt{x+4}$$
$$\left(\frac{x+1}{2}\right)^2 = x + 4$$
$$\frac{(x+1)(x+1)}{(2)(2)} = x + 4$$
$$\frac{x^2 + 2x + 1}{4} = x + 4$$
$$x^2 + 2x + 1 = 4(x+4)$$
$$x^2 + 2x + 1 = 4x + 16$$
$$x^2 - 2x - 15 = 0$$
$$(x-5)(x+3) = 0$$
$$x = 5\ and\ x = -3$$

Domain Restrictions

$$x + 4 \geq 0$$
$$x \geq -4$$

Both answers are consistent with the restrictions

Check for extraneous answers

$$x = 2\sqrt{x+4} - 1$$
$$5 = 2\sqrt{5+4} - 1$$
$$5 = 6 - 1$$
$$5 = 5$$

True

When $x = 5, y = x$ so $y = 5$ intersection: $(5, 5)$

$$x = 2\sqrt{x+4} - 1$$
$$-3 = 2\sqrt{-3+4} - 1$$
$$-3 = 2 - 1$$
$$-3 = 1$$

False

$$\therefore The\ functions\ intersect\ at\ the\ point\ (5,5)$$

Solving Square Root Inequalities

To solve a square root inequality:

1) Isolate the radical
2) State the domain restrictions
3) Solve (remember to change the inequality if you multiply or divide by a negative number)
4) Check against domain restrictions
5) State solution

Ex: Solve $2\sqrt{2x+4}-6>2$

$$2\sqrt{2x+4}-6>2$$
$$2\sqrt{2x+4}>8$$
$$\sqrt{2x+4}>4$$
$$2x+4>16$$
$$2x>12$$
$$x>6$$

Domain Restrictions

$$2x+4\geq 0$$
$$2x\geq -4$$
$$x\geq -2$$

Check against domain restrictions

The solution is $x>6$. That is consistent with the domain restrictions, and the domain offers no additional restrictions.

Solution: $x>6$ or $2\sqrt{2x+4}-6>2\ over\ ]6,+\infty[$

Ex: Solve $-2\sqrt{x-3}+4>0$

$$-2\sqrt{x-3}+4>0$$
$$-2\sqrt{x-3}>-4$$
$$\sqrt{x-3}<2$$
$$x-3<4$$
$$x<7$$

Domain Restrictions

$$x-3\geq 0$$
$$x\geq 3$$

Check against domain restrictions

This domain offers us a lower limit of 3, and our solution gives us an upper limit of 7.

Solution: $3\leq x<7$
or $-2\sqrt{x-3}+4>0\ over\ [3,7[$

Ex: Solve $2\sqrt{3(x-4)}+2 \geq 1$

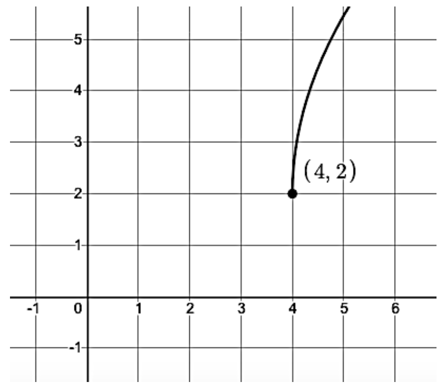
$$2\sqrt{3(x-4)}+2 \geq 1$$
$$2\sqrt{3(x-4)} \geq -1$$
$$\sqrt{3(x-4)} \geq -0.5$$

We know the term under the radical cannot equal a negative number, so we either have no solution or a solution that exists everywhere the function is defined.

Check a sketch to determine solution

Sketch

Vertex: $(4, 2) \quad a > 0 \quad b > 0$



From the sketch we can see that the function is always greater than or equal to 1 (but limited to the domain of the function)

Solution: $4 \leq x$

or $2\sqrt{3(x-4)}+2 \geq 1 \; over \; [4, +\infty[$

Finding the Inverse of a Square Root Function

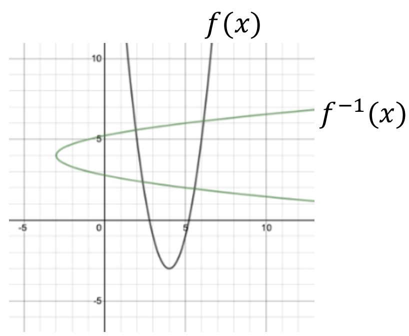
We find the inverse of a square root function in the same way we find the inverse of any function.

Important note: we can think of a square root function as half of a sideways parabola. Because of this, square root functions and squared functions (parabolas) are not the exact inverse of each other – there are domain/range restrictions.

Recall: the domain of a function becomes the range of its inverse and the range of a function becomes the domain of its inverse.

Ex: Find the inverse of $f(x) = 2(x-4)^2 - 3$,sketch both, and state the domain and range for both.

$y = 2(x-4)^2 - 3$ Domain: $]-\infty, +\infty[$ Range: $[-3, +\infty[$



Inverse

$x = 2(y-4)^2 - 3$

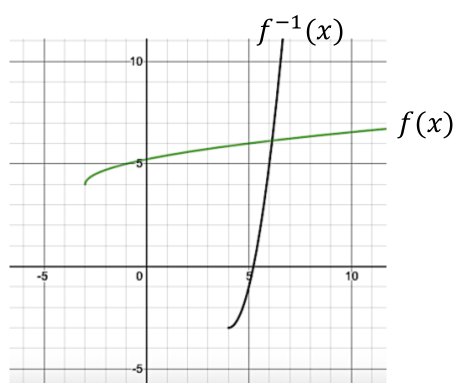
$$y = \sqrt{\frac{x+3}{2}} \quad so\ f^{-1}(x) = \sqrt{\frac{x+3}{2}}$$

Domain: $[-3, +\infty[$ Range: $]-\infty, +\infty[$

Note that the inverse is a relation (not a function) because we have to consider the positive and negative solutions when taking the square root, which is not the case in a square root function.

Ex: Find the inverse of $y = \sqrt{\frac{x+3}{2}} + 4$, sketch both, and state the domain and range for both.

$y = \sqrt{\frac{x+3}{2}} + 4$ Domain: $]-3, +\infty[$ Range: $[4, +\infty[$



Inverse

$x = \sqrt{\frac{y+3}{2}} + 4$

$y = 2(x-4)^2 - 3$ $so\ f^{-1}(x) = 2(x-4)^2 - 3$

Domain: $[4, +\infty[$ Range: $]-3, +\infty[$

Note the restricted domain, which gives us only half of a parabola. This is because when talking about square root functions, we only take the positive root, by definition.