Rational Functions

Rational Functions Basics

- The graph of a rational function is formed by two symmetrical curves and two asymptotes
- Each curve gets infinitely close to each asymptote (but never touches or crosses it)
- Domain: $]-\infty, h[\ \cup\ ]h, +\infty[$
- Range: $]-\infty, k[\ \cup\ ]k, +\infty[$
- The inverse of a rational function is another rational function

There are two ways to write the rule of a rational function.

$$\text{Standard Form: } f(x) = \frac{a}{b(x-h)} + k$$

$$\text{where } a \neq 0 \text{ and } b \neq 0 \text{ and asymptotes at } x = h \text{ and } y = k$$

- If a and b have same signs, curves are in quadrants 1 and 3
- If a and b have opposite signs, curves are in quadrants 2 and 4

$$\text{General Form: } f(x) = \frac{a_1x+b_1}{a_2x+b_2}$$

$$\text{asymptotes at } x = -\frac{b_2}{a_2} \text{ and } y = \frac{a_1}{a_2}$$

- Need to solve for x and/or y intercepts to determine location of curves.

Expressing rational functions in different forms

Ex: Removing b

(you can always do this, even if it means a is expressed as a fraction or decimal)

$$f(x) = \frac{8}{2x-20} + 5$$

$$f(x) = \frac{8}{2(x-10)} + 5$$

$$f(x) = \frac{4}{x-10} + 5$$

Ex: Changing from general form to standard form

(you never actually need to do this, since we can determine the asymptotes from both forms)

$$f(x) = \frac{3x+5}{x-1}$$

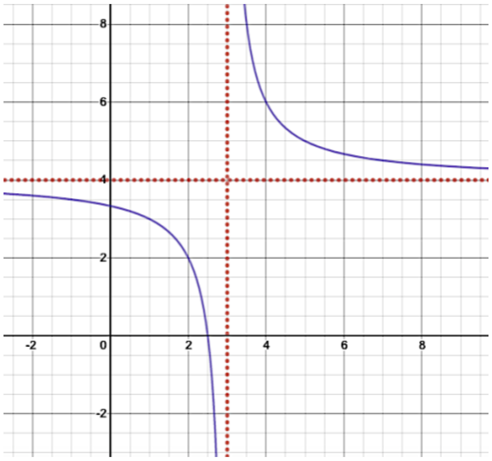
Use polynomial long division

$$\begin{array}{r} 3 \\ x-1\,\big|\overline{3x+5} \\ \underline{-(3x-3)} \\ 8 \end{array} \quad \rightarrow \quad \frac{8}{x-1} + 3$$

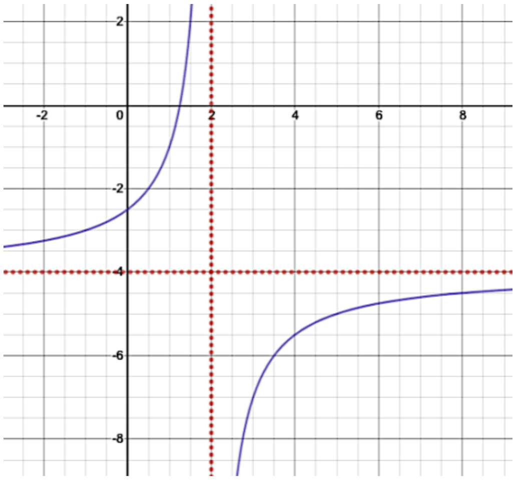
Sketching a Rational Function

1) Determine the asymptotes and draw them
2) Determine location of curves (based on signs of a and b or by solving for x and/or y intercept)
3) Sketch function

Ex: Sketch the function $f(x) = \frac{2}{x-3} + 4$
Asymptotes at $x = 3$ and $y = 4$
a and b are both positive, so curves in quadrants 1 and 3



Ex: Sketch the function $f(x) = -\frac{3}{x-2} - 4$
Asymptotes at $x = 2$ and $y = -4$
a is negative and b is positive so curves in quadrants 2 and 4



Ex: Sketch the function $f(x) = \frac{-4x+2}{2x-2}$

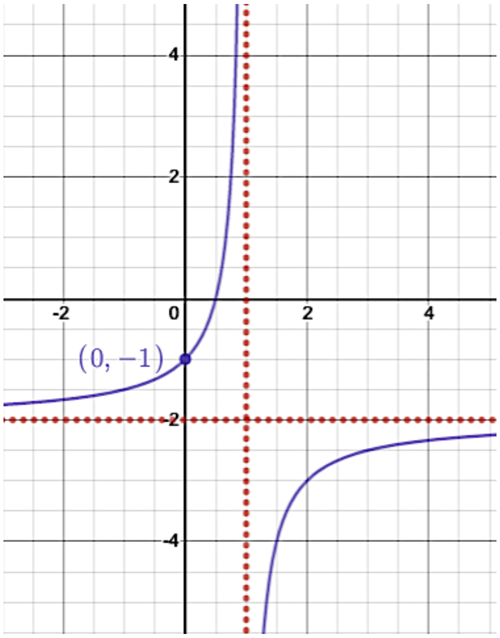
Asymptotes at $y = \frac{a_1}{a_2}$

$$x = -\frac{b_2}{a_2} = -\frac{-2}{2} = 1$$

$$y = \frac{a_1}{a_2} = \frac{-4}{2} = -2$$

y-intercept ($x = 0$)

$$f(x) = \frac{-4x+2}{2x-2} = \frac{-4(0)+2}{2(0)-2} = -1$$



When we find asymptotes, we need to make sure the values do not make both the numerator and the denominator 0 at the same time.

Ex: $f(x) = \frac{4x+2}{2x+1}$

Asymptotes:

$x = -\frac{1}{2}$ and $y = \frac{4}{2} = 2$

Let's check $x = -\frac{1}{2}$

$$f(x) = \frac{4x+2}{2x+1}$$

$$f(x) = \frac{4\left(-\frac{1}{2}\right)+2}{2\left(-\frac{1}{2}\right)+1}$$

$$f(x) = \frac{-2+2}{-1+1}$$

$$f(x) = \frac{0}{0} \ \ THIS\ IS\ A\ PROBLEM!$$

Let's do some polynomial long division to see what's going on.

$$\begin{array}{r} 2 \\ 2x+1\overline{)4x+2} \\ -\underline{(4x+2)} \end{array}$$

$$0 \rightarrow \frac{0}{2x+1} + 2 \rightarrow 2$$

This means $f(x) = 2$, which is a horizontal line with a hole at $x = -\frac{1}{2}$

Finding the Rule of a Rational Function

From a graph (or given asymptotes and a point):

1) Substitute in asymptotes (h and k)
2) Substitute in point (x and y)
3) Solve for a
4) Write the function (and get rid of decimal/fraction in numerator if it exists)

Ex: Find the rule of a rational function with asymptotes at $y = 5$ and $x = 2$ and passing through the point $(3.5, 6)$

$$y = \frac{a}{x-h} + k$$

$$6 = \frac{a}{3.5-2} + 5$$

$$1 = \frac{a}{1.5}$$

$$1.5 = a$$

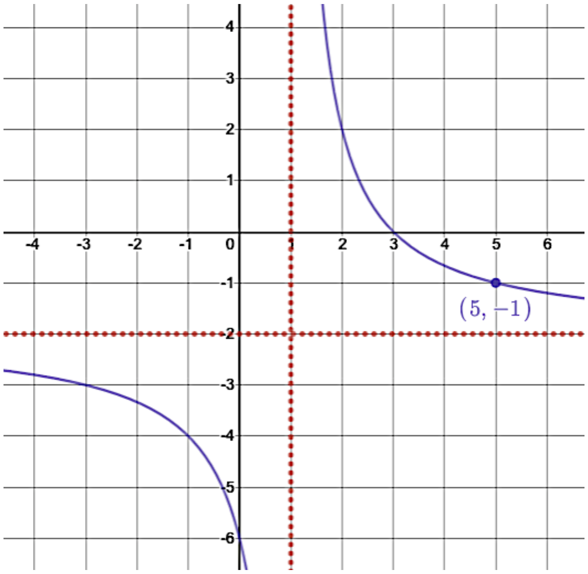
$$y = \frac{1.5}{x-2} + 5$$

$$y = \left(\frac{2}{2}\right)\frac{1.5}{x-2} + 5$$

Multiply by 1 in order to eliminate decimal/fraction in numerator.

$$\therefore y = \frac{3}{2(x-2)} + 5$$

Ex: Find the rule of the rational function graphed below.



$h = 1$ and $k = -2$

$$y = \frac{a}{x-h} + k$$

$$-1 = \frac{a}{5-1} - 2$$

$$1 = \frac{a}{4}$$

$$4 = a$$

$$\therefore f(x) = \frac{4}{x-1} - 2$$

We can also find the rule of rational functions from words.

Ex: To enter a basketball tournament, players much pay their share of the \$500 transport cost, plus an additional \$20 feel. Write a rule that will determine the amount each student will pay based on the number of players attending.

The amount each player pays is the total cost divided by the number of players.
Total cost: $500 + 20x$
Number of players: x

$$\therefore \, Each \; player \; pays \frac{500 + 20x}{x}$$

Ex: The school holds a dance and charges \$10 per student to attend. The cost of a DJ is \$500. Write a rule associating profits made per student in attendance.

The profit per student is the total profit divided by the number of students
Total profit: $10x - 500$
Number of students: x

$$\therefore \, Profit \; per \; student \; is \; \frac{10x - 500}{x}$$

3. During its first year as a member of a community-supported agriculture network, a farm includes 25 families and its operating costs are \$16 250. Every year after that first year, the number of member families increases by 2 and the operating costs increase by \$800. Write a rule that allows you to determine the distribution of operating costs between families in relation to the number of years since the farm joined the network.

Let x be the number of years since the farm joined the network.
The rule needs to be the total costs divided by the number of families.
Total costs: $16250 + 800x$
Number of families: $25 + 2x$

$$\therefore \, Cost \; per \; family \; is \; \frac{16250 + 800x}{25 + 2x}$$

4. 10g of substance A are placed in 100mL of water. Every minute 2 g of substance A is added and 5mL of water is added. Write a rule that allows you to determine the concentration of substance A over time, in minutes (that is, the amount of substance A compared with the amount of water).

Let x be the number of minutes after initial mixing
Substance A: $10 + 2x$
Water: $100 + 5x$

$$\therefore \, Concentration \; is \; \frac{10 + 2x}{100 + 5x}$$

Solving Rational Functions

To solve for y given x, plug in x and solve for y.

Ex: Given $f(x) = \frac{4}{2x+3} - 7, \text{solve for } f(2)$

$$f(x) = \frac{4}{2x+3} - 7$$

$$f(2) = \frac{4}{2(2)+3} - 7$$

$$f(2) = \frac{4}{4+3} - 7$$

$$f(2) = \frac{4}{7} - 7$$

$$f(x) = -\frac{45}{7} \; or - 6.4286$$

To solve for x given y:

1) Plug in y and solve for x
2) Check restrictions (denominator cannot equal 0)
3) Always consult the scenario for appropriate answers when solving word problems

Ex: Given $f(x) = \frac{3}{2x+4} - 6$, find the zero

$$f(x) = \frac{3}{2x+4} - 6$$

$$0 = \frac{3}{2x+4} - 6$$

$$6 = \frac{3}{2x+4}$$

$$6(2x+4) = 3$$

$$12x + 24 = 3$$

$$12x = -21$$

$$x = -\frac{21}{12} \; or - 1.75$$

Restrictions: $2x + 4$ cannot equal 0

$$2x + 4 \neq 0$$

$$2x \neq -4$$

$$x \neq -2$$

Since $-1.75 \neq 2$

$x = -1.75 \; is \; the \; solution$

Ex: Given $f(x) = \frac{3x+4}{2x-1}$, solve for x when $f(x) = 4$

$$f(x) = \frac{3x+4}{2x-1}$$

$$4 = \frac{3x+4}{2x-1}$$

$$4(2x-1) = 3x+4$$

$$8x - 4 = 3x + 4$$

$$5x = 8$$

$$x = \frac{8}{5}$$

Restrictions: $2x - 1$ cannot equal 0

$$2x - 1 \neq 0$$

$$2x \neq 1$$

$$x \neq \frac{1}{2}$$

Since $\frac{8}{5} \neq \frac{1}{2}$

$x = \frac{8}{5}$ *is the solution*

Ex: In an experiment, ten mL of substance A are placed in 110 mL of water. Every minute, 2 mL of substance A and 7 mL of water are added.

Given concentration is measured as $\frac{amount\ of\ substance\ A}{total\ amount\ (water+substance\ A)}$,

a) when will the concentration be 0.02 (or 2%)?

Substance A: $10 + 2x$

Water: $110 + 7x$

Total: $10 + 2x + 110 + 7x$

$$120 + 9x$$

$$Concentration = \frac{10+2x}{120+9x}$$

$$0.02 = \frac{10+2x}{120+9x}$$

$$0.02(120 + 9x) = 10 \mp 2x$$

$$2.4 + 0.18x = 10 + 2x$$

$$-1.82x = 7.6$$

$$x = -4.1758$$

Restrictions: $120 + 9x$ cannot equal 0

$$120 + 9x \neq 0$$

$$9x \neq -120$$

$$x \neq 13.3\overline{3}$$

a) However, in the question, x cannot be negative (it is time, in minutes, since the experiment began), therefore, there is no solution.

Solving Rational Inequalities

To solve an inequality:

1) change the inequality to an equality
2) solve
3) consult the graph to determine the answer

Ex: Given $f(x) = \frac{2}{x-3} + 2$, determine the interval over which $f(x) > 0$

$$\frac{2}{x-3} + 2 > 0$$

$$\frac{2}{x-3} + 2 = 0$$

$$\frac{2}{x-3} = -2$$

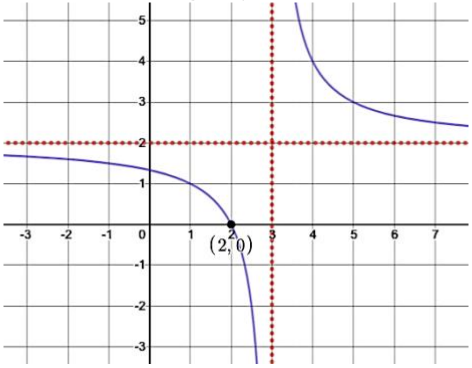
$$2 = -2(x-3)$$

$$2 = -2x + 6$$

$$-4 = -2x$$

$$2 = x$$

Sketch:
Asymptotes: $x = 3$ and $2 = 2$
a and b same sign (quadrants 1 and 3)



$$\therefore \frac{2}{x-3} + 2 > 0\ over\ ]-\infty, 2[\ \cup\ ]3, +\infty[$$

Ex: Determine the interval over which $4 \geq \frac{2x+6}{8-3x}$

$$4 = \frac{2x+6}{8-3x}$$

$$4(8-3x) = 2x + 6$$

$$32 - 12x = 2x + 6$$

$$-14x = -26$$

$$x = \frac{13}{7}$$

Asymptotes:

$$f(x) = \frac{2x+6}{8-3x} = \frac{2x+6}{-3x+8}$$

$$y = -\frac{2}{3} \text{ and } x = \frac{8}{3}$$

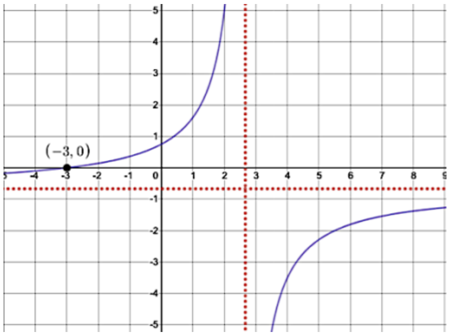
Zero:

$$0 = \frac{2x+6}{8-3x}$$

$$0 = 2x + 6$$

$$-6 = 2x$$

$$-3 = x$$



$$\therefore 4 \geq \frac{2x+6}{8-3x}\ over\ \left]-\infty, \frac{13}{7}\right[\ \cup\ \left]\frac{8}{3}, +\infty\right[$$

Finding the Inverse of Rational Functions

Remember:

To find the inverse of a rational function, swap the x and y and solve for y.

The domain of $f(x)$ becomes the range of $f^{-1}(x)$

The range of $f(x)$ becomes the domain of $f^{-1}(x)$

These mean that the horizontal asymptote and the vertical asymptote switch.

Ex: Find the inverse of the following functions:

$a)\ f(x) = \dfrac{2x+4}{3x-8}$

$$x = \frac{2y+4}{3y-8}$$

$$x(3y-8) = 2y+4$$

$$3xy - 8x = 2y + 4$$

$$3xy - 2y = 8x + 4$$

$$y(3x-2) = 8x + 4$$

$$y = \frac{8x+4}{3x-2}$$

$$\therefore f^{-1}(x) = \frac{8x+4}{3x-2}$$

$b)\ g(x) = \dfrac{4}{3x-8} + 2$

$$x = \frac{4}{3y-8} + 2$$

$$x - 2 = \frac{4}{3y-8}$$

$$(x-2)(3y-8) = 4$$

$$3y - 8 = \frac{4}{x-2}$$

$$3y = \frac{4}{x-2} + 8$$

$$y = \frac{4}{3(x-2)} + \frac{8}{3}$$

$$\therefore g^{-1}(x) = \frac{4}{3(x-2)} + \frac{8}{3}$$